Weak Convergence Approach to a Parabolic Equation with Large Random Potential

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Abstract

Solutions to partial differential equations with highly oscillatory, large random potential have been shown to converge either to homogenized, deterministic limits or to stochastic limits depending on the statistical properties of the potential. In this paper we consider a large class of piece-wise constant potentials and precisely describe how the limit depends on the the correlation properties of the potential and on spatial dimension $d \geq 3$. The derivations are based on a Feynman-Kac probabilistic representation and on an invariance principle for Brownian motion in a random scenery.

1 Introduction

Partial differential equations with small scale structures arise in many aspects of physics and applied science, and homogenization theory has proved to be useful both from the theoretical and numerical point of view. Depending on the statistical property of the small scale coefficients, the limit, as the size of the fluctuations tends to zero, could be either deterministic or random. Understanding the dependence of the limiting equation on the coefficients is therefore important in many practical settings.

Parabolic equations (including heat Anderson or Schrödinger models) with large, random Gaussian potential are analyzed in [1, 2, 3, 19, 20] using Duhamel infinite series expansions and combinatorial techniques. The limiting equation is either a deterministic, homogenized, parabolic equation or a stochastic partial differential equation depending on the correlation properties of the (possibly time-dependent) potential; see [3]. In [16, 9], a one dimensional heat equation with singular time-dependent coefficients is studied by probabilistic (Feynman-Kac formula) and analytical techniques, respectively. The limiting equation is also deterministic or random depending on whether the coefficients have microstructure in both temporal and spatial variables. In this paper, we follow the Feynman-Kac approach to analyze a parabolic equation in dimension $d \geq 3$ of the form $\partial_t u_\varepsilon = \Delta u_\varepsilon + iV_\varepsilon u_\varepsilon$, where V_ε is a large, time-independent, random potential, which is not necessarily Gaussian. An imaginary potential is introduced to obtain a uniform bound on the energy

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of the solution, which considerably helps in the analysis of exponential functionals of Brownian motion and of the limit as $\varepsilon \to 0$. The corresponding heat equation (with iV_ε replaced by V_ε) might be analyzed using techniques in [9] but is not considered here. The above scalar equation may be recast as the system

$$\partial_t \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \Delta & 0 \\ 0 & \Delta \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} 0 & -V_\varepsilon \\ V_\varepsilon & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \tag{1.1}$$

in which V_ε might model a conservative process of interaction between two populations and find applications in biological or social sciences.

The Feynman-Kac representation is applied in [15] to a one-dimensional heat equation with time-independent coefficients. Using the same probabilistic representation, as well as auxiliary problems, [14] addresses homogenization of divergence-form operators with lower-order random potentials written in terms of gradients of stationary processes. In [10], the authors study the Feynman-Kac formula for a heat equation driven by a fractional white noise in all spatial dimensions. This corresponds to V_ε with a sufficiently long-range correlation function (i.e., sufficiently slow decay). In [13], the heat equation with long-range correlated Gaussian potential is studied with a similar type of limiting equation as in [10].

If we consider the random integral appearing in the Feynman-Kac formula, i.e., $\int_0^t V_\varepsilon(B_s)ds$, where V_ε is the random potential, and B_s is some independent Brownian motion, we obtain the so-called problem of Brownian motion in a random scenery. In the discrete case, this corresponds to the Kesten-Spitzer model of random walk in a random scenery. An invariance principle has been proved in [11, 6] in all spatial dimensions. A continuous version for a specific type of potential has been analyzed in [17], primarily for $d = 1, 2$. In [12], the authors analyze the case where the Brownian motion is replaced by some reversible stationary Markov process and prove a central limit result. In the aforementioned examples, the random scenery is independent of the random walk. This is different from the so-called random walk in random environment, where the underlying environment indeed affects the random walk, e.g., in the random conductance and percolation clusters model. For Brownian motion in random scenery, in addition to central limit theorem, most of the existing results are of large deviation type, and we refer to [18] and references therein.

In [7], we prove an invariance principle for $\int_0^t V_\varepsilon(B_s)ds$ in all dimensions when the potential is stationary Gaussian or Poissonian. Although we do not do it here, the limiting results obtained in this paper can be extended to those case without much effort. Instead, we adopt the specific class of potentials introduced in [17], which allows us to treat the short-range and long-range cases in a uniform way, and prove a weak convergence result when $d \geq 3$ and obtain the asymptotic limit of our PDE solution. As mentioned above, the limiting equation turns to be either deterministic or random depending on the correlation properties of the potential V_ε.

2 Problem setup

We are interested in the parabolic equation

$$\partial_t u_\varepsilon = \frac{1}{2}\Delta u_\varepsilon + i\frac{1}{\varepsilon^\beta} V(\frac{x}{\varepsilon}) u_\varepsilon \tag{2.1}$$

with initial condition $u_\varepsilon(0,x) = f(x)$, $x \in \mathbb{R}^d$, $d \geq 3$. Here V is a stationary random field with either short- or long-range correlation, which will be defined later. We wish to analyze the limiting behavior of u_ε as $\varepsilon \to 0$ and its dependence on the random field. It turns out that as long as the covariance function of the random field does not decay too slowly, then homogenization and convergence to a deterministic solution occurs, while when the random fields are sufficiently long-range, the limiting equation is a stochastic PDE. Correspondingly, the large, highly oscillatory random field $\frac{1}{\varepsilon^\beta}V(\frac{x}{\varepsilon})$ is either replaced by a constant or by a Gaussian noise, respectively, in the limiting equation.

The random field we consider is of the form $V(x) = \xi_{[x+U]}$, where $\{\xi_k, k \in \mathbb{Z}^d\}$ is a sequence of stationary random variables, U is uniformly distributed in $[0,1)^d$ and independent of $\{\xi_k\}$, and $[x] := ([x_1], ..., [x_d])$ with $[x_i]$ denoting the largest integer smaller than or equal to x_i. Defining $r(k) = \mathbb{E}\{\xi_0\xi_k\}$, we verify that $R(x) := \mathbb{E}\{V(y)V(y+x)\} = \sum_k r(k)\prod_{i=1}^d (1 - |x_i - k_i|)1_{|x_i-k_i|<1}$, and thus, the decay of $r(k)$ as $|k| \to \infty$ determines the decay of $R(x)$ as $|x| \to \infty$.

The three cases of ξ_k we consider are the following:

1. *Short-range correlation*: $\{\xi_k\}$ are i.i.d random variables with zero mean and unit variance, and for technical reasons we assume $\mathbb{E}\{\xi_0^6\} < \infty$. In this case,

$$R(x) = \prod_{i=1}^d (1 - |x_i|)1_{|x_i|<1}. \tag{2.2}$$

2. *Long-range correlation*:

- Case 1: $\{\xi_k\}$ is a stationary Gaussian sequence with zero mean and unit variance. We assume its covariance function

$$r(k) = \mathbb{E}\{\xi_0\xi_k\} \sim \frac{c_d}{|k|^\alpha} \tag{2.3}$$

for some $\alpha \in (2, d)$ as $|k| \to \infty$. In this case,

$$R(x) = \sum_k r(k) \prod_{i=1}^d (1 - |x_i - k_i|)1_{|x_i-k_i|<1} \sim \frac{c_d}{|x|^\alpha} \tag{2.4}$$

as $|x| \to \infty$.

- Case 2: $\{\xi_k = \Phi(\eta_k)\}$, where $\{\eta_k\}$ is a stationary Gaussian sequence with zero mean and unit variance. We assume its covariance function $\rho(k) = \mathbb{E}\{\eta_0\eta_k\}$ satisfies $|\rho(k)| \leq C\prod_{i=1}^d \min(1, \frac{1}{|k_i|^{\alpha_i}})$ for some constant C and

$$\rho(k) \sim \frac{c_d}{\prod_{i=1}^d |k_i|^{\alpha_i}} \tag{2.5}$$

as $\min_{i=1,...,d} |k_i| \to \infty$ for some $\alpha_i \in (0,1)$ and constant $c_d > 0$. The function Φ satisfies $\mathbb{E}\{\Phi(\eta_k)^2\} < \infty, V_0 = \mathbb{E}\{\Phi(\eta_k)\} = 0$ and $V_1 = \mathbb{E}\{\Phi(\eta_k)\eta_k\} \neq 0$. We further assume that $\alpha := \sum_{i=1}^d \alpha_i \in (0,2)$.

Since $r(k) = \mathbb{E}\{\xi_0 \xi_k\}$, we will see that $r(k) \sim V_1^2 \rho(k)$ as $|k| \to \infty$, so

$$R(x) = \sum_k r(k) \prod_{i=1}^{d} (1 - |x_i - k_i|) 1_{|x_i - k_i| < 1} \sim \frac{V_1^2 c_d}{\prod_{i=1}^d |x_i|^{\alpha_i}} \tag{2.6}$$

as $\min_{i=1,\dots,d} |x_i| \to \infty$.

Remark 2.1. In the short-range case, the assumption $\mathbb{E}\{\xi_0^6\} < \infty$ is used in proving the tightness results of a family of processes in $\mathcal{C}([0,T])$, which we do not necessarily need in the proof of Theorem 3.11.

Remark 2.2. In both long-range correlation cases, we assume $\alpha < d$, which implies that $R(x)$ is not integrable.

In the first two cases, we define the positive constant that will appear in the homogenized equation

$$\sigma_d = \left(\frac{4}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{\hat{R}(\xi)}{|\xi|^2} d\xi \right)^{\frac{1}{2}} = \left(\frac{1}{\pi^{\frac{d}{2}}} \Gamma(\frac{d}{2} - 1) \int_{\mathbb{R}^d} \frac{R(x)}{|x|^{d-2}} dx \right)^{\frac{1}{2}} > 0 \tag{2.7}$$

with $\hat{R}(\xi) = \int_{\mathbb{R}^d} R(x) e^{-i\xi x} dx$.

The scaling constant β is defined as

$$\beta = \begin{cases} 1, & \text{short-range correlation, long-range correlation case 1,} \\ \frac{\alpha}{2}, & \text{long-range correlation case 2.} \end{cases} \tag{2.8}$$

In the short-range correlation case, the random field $V(x)$ is mixing in the following sense. For two Borel sets $A, B \subset \mathbb{R}^d$, we denote by $\mathcal{F}_A$ and $\mathcal{F}_B$ the sub-σ algebras generated by the field $V(x)$ for $x \in A$ and $x \in B$, respectively. There exists a mixing coefficient $\phi(r)$ such that

$$|Cor(\eta, \zeta)| \leq \phi(2d(A,B)) \tag{2.9}$$

for all square integrable random variables η and ζ that are $\mathcal{F}_A$ and $\mathcal{F}_B$ measurable respectively. The multiplicative factor 2 is only here for convenience. In the short-range correlation case, when $|x|$ is large enough, $V(x+y)$ is independent from $V(y)$, so the mixing coefficient $\phi(r)$ can be chosen as a positive, decaying function with compact support in $[0,\infty)$. We will use this in the calculation of the fourth moment of $V(x)$.

We also note that since the potential $V(x)$ is discontinuous and possibly unbounded, (2.1) is not always solvable in a classical sense. We show that the solution defined by the Feynman-Kac formula is a weak solution in the following proposition.

Proposition 2.3. *Consider the equation*

$$u_t = \frac{1}{2} \Delta u + iV(x) u \tag{2.10}$$

with initial condition $u(0,x) = f(x) \in \mathcal{C}_b(\mathbb{R}^d)$. *Let us define*

$$u(t,x) = \mathbb{E}_B\{f(x + B_t) \exp(i \int_0^t V(x + B_s) ds)\}, \tag{2.11}$$

where $\mathbb{E}_B$ denotes the expectation with respect to the Brownian motion starting from the origin.

Then for any $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$, we have

$$\int_{\mathbb{R}^d} u(t,x)\varphi(x)dx = \int_{\mathbb{R}^d} f(x)\varphi(x)dx + \int_0^t \int_{\mathbb{R}^d} u(s,x)\frac{1}{2}\Delta\varphi(x)dxds + \int_0^t \int_{\mathbb{R}^d} u(s,x)V(x)\varphi(x)dxds \quad (2.12)$$

almost surely, i.e., the Feynman-Kac solution $u(t,x)$ is a weak solution to (2.10) almost surely.

Proof. Recall $V(x) = \xi_{[x+U]}$, fix any $\delta, M > 0$, define

$$V_{\delta,M}(x) = \int_{\mathbb{R}^d} \phi_\delta(x-y)V(y)1_{|y|<M}dy, \quad (2.13)$$

where ϕ_δ is a family of compactly supported mollifier. For each realization, since $V(y)1_{|y|<M}$ is bounded, $V_{\delta,M}$ is bounded and as long as V is continuous at x, we have $V_{\delta,M}(x) \to V(x)$ as $\delta \to 0, M \to \infty$. In addition, $V_{\delta,M}$ is smooth, so for the equation $\partial_t u_{\delta,M} = \frac{1}{2}\Delta u_{\delta,M} + V_{\delta,M}u_{\delta,M}$ with initial condition $u_{\delta,M}(0,x) = f(x)$, we have its classical solution given by the Feynman-Kac formula

$$u_{\delta,M}(t,x) = \mathbb{E}_B\{f(x+B_t)\exp(i\int_0^t V_{\delta,M}(x+B_s)ds)\}, \quad (2.14)$$

and it is straightforward to check that $u_{\delta,M}(t,x) \to u(t,x)$ as $\delta \to 0, M \to \infty$ by Lebesgue dominated convergence theorem. Since $u_{\delta,M}$ is also the weak solution, we have

$$\begin{aligned}\int_{\mathbb{R}^d} u_{\delta,M}(t,x)\varphi(x)dx \quad = \quad & \int_{\mathbb{R}^d} f(x)\varphi(x)dx + \int_0^t \int_{\mathbb{R}^d} u_{\delta,M}(s,x)\frac{1}{2}\Delta\varphi(x)dxds \\ & + \int_0^t \int_{\mathbb{R}^d} u_{\delta,M}(s,x)V_{\delta,M}(x)\varphi(x)dxds.\end{aligned} \quad (2.15)$$

Let $\delta \to 0, M \to \infty$, we finish the proof. □

Since $V(x)$ may be unbounded, proving uniqueness of the solution to (2.10) is a difficult task. Such a task becomes easy when the equation is posed on a bounded domain since V is then bounded almost surely. But calculations with the corresponding Brownian motion on bounded domains involve standard complications which we wish to avoid here. When we refer to "the" solution to (2.1), we therefore mean the weak solution given by the Feynman-Kac probabilistic representation in the rest of the paper.

In the following sections, we deal with the short-range correlation and long-range correlation cases in turn. Let us introduce some notation. We have randomness coming from two sources, the random potential V from the differential equation and the Brownian motion from the Feynman-Kac formula, and consider the usual product probability space. We use $B = \{B_t\}$ to denote the Brownian motion starting from the origin and $\mathbb{E}_B$ is used to denote the expectation only with respect to B. The expectation in the whole product probability space is denoted by $\mathbb{E}$. $N(\mu,\sigma^2)$ is the Gaussian distribution with mean μ and variance σ^2, and $q_t(x)$ denotes the density function of $N(0,t)$. We use $a \wedge b = \min(a,b)$ and $a \lesssim b$ when there exists a constant C such that $a \leq Cb$.

3 Short-range correlation

Since we rely on the Feynman-Kac formula, the integral $\frac{1}{\varepsilon}\int_0^t V(\frac{B_s}{\varepsilon})ds$ invites us to look at Brownian motion in a random scenary. Heuristically, $V(x)$ is piecewise constant for each realization, so the integral is a sum of independent random variables, and we would expect a central limit theorem to hold.

We first prove an invariance principle for Brownian motion in a random scenery. Using that weak convergence result together with the a priori bound for u_ε, we pass to the limit as $\varepsilon \to 0$ to derive the homogenized equation.

3.1 Invariance principle for Brownian motion in a random scenery when $d \geq 3$

Define $X_\varepsilon(t) = \frac{1}{\varepsilon}\int_0^t V(\frac{B_s}{\varepsilon})ds$, the main result is the following theorem.

Theorem 3.1. *In the short-range correlation case,*

$$X_\varepsilon(t) \Rightarrow \sigma_d W_t \tag{3.1}$$

weakly in $\mathcal{C}([0,T])$, where W_t is a standard Brownian motion.

To prove it, by scaling property of Brownian motion, we know $X_\varepsilon(t)$ has the same distribution as $\varepsilon\int_0^{t/\varepsilon^2} V(B_s)ds$, and defining

$$p_t(k) = \int_0^t 1_{[B_s+U]=k}ds, \tag{3.2}$$

it remains to prove the weak convergence of $X_\varepsilon(t) = \varepsilon\sum_k \xi_k p_{t/\varepsilon^2}(k)$. This is divided into two steps. We first prove the convergence of finite dimensional distributions and then prove tightness. In the following, we do not distinguish between $\frac{1}{\varepsilon}\int_0^t V(\frac{B_s}{\varepsilon})ds$ and $\varepsilon\int_0^{t/\varepsilon^2} V(B_s)ds$.

3.1.1 Finite dimensional distributions

Let $X(t) = \int_0^t V(B_s)ds = \sum_k \xi_k p_t(k)$, we first prove two lemmas, which combined show that the variance of $X_\varepsilon(t)$ conditioning on B, U converges in L^2.

Lemma 3.2. $\mathbb{E}\{\sum_k p_t(k)^2\} \sim \sigma_d^2 t$.

Lemma 3.3. $\mathbb{E}\{(\sum_k p_t(k)^2)^2\} \sim \sigma_d^4 t^2$.

Proof of Lemma 3.2. Since $\mathbb{E}\{\sum_k p_t(k)^2\} = \mathbb{E}\{X(t)^2\}$, we have

$$\begin{aligned}
\mathbb{E}\{\sum_k p_t(k)^2\} =& \mathbb{E}\{(\int_0^t V(B_s)ds)^2\} = \int_0^t\int_0^t \mathbb{E}\{R(B_s - B_u)\}dsdu \\
=& \frac{1}{(2\pi)^d}\int_0^t\int_0^t\int_{\mathbb{R}^d} \mathbb{E}\{\exp(i\xi(B_s - B_u))\}\hat{R}(\xi)d\xi dsdu \\
=& \frac{1}{(2\pi)^d}\int_0^t\int_0^t\int_{\mathbb{R}^d} \exp(-\frac{1}{2}|\xi|^2|s-u|)\hat{R}(\xi)d\xi dsdu \\
=& \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\hat{R}(\xi)\frac{4}{|\xi|^2}\left(t - \frac{2}{|\xi|^2}(1 - e^{-|\xi|^2 t/2})\right)d\xi.
\end{aligned} \tag{3.3}$$

Because $d \geq 3$, $\hat{R}(\xi)/|\xi|^2$ is integrable, then by the Lebesgue dominated convergence theorem, we have

$$\mathbb{E}\{\sum_k p_t(k)^2\}/t \to \frac{4}{(2\pi)^d}\int_{\mathbb{R}^d} \frac{\hat{R}(\xi)}{|\xi|^2} d\xi$$

as $t \to \infty$. $\square$

Proof of Lemma 3.3. Since $p_t(k)$ is independent of the choice of ξ_k, we can assume that ξ_k have Gaussian distribution, in which case we have $\mathbb{E}\{(\sum_k p_t(k)^2)^2\} = \frac{1}{3}\mathbb{E}\{X(t)^4\}$, and the fourth moment can be calculated as

$$\mathbb{E}\{X(t)^4\} = 24\int_{S_t} \mathbb{E}\{\prod_{i=1}^4 V(B_{s_i})\} ds, \tag{3.4}$$

where we denote $S_t = \{0 < s_1 < ... < s_4 < t\}$. It turns out that

$$\begin{aligned}\mathbb{E}\prod_{i=1}^4 V(x_i) =& \mathbb{P}([x_1+U] = [x_2+U], [x_3+U] = [x_4+U]) \\ &+ \mathbb{P}([x_1+U] = [x_3+U], [x_2+U] = [x_4+U]) \\ &+ \mathbb{P}([x_1+U] = [x_4+U], [x_2+U] = [x_3+U]),\end{aligned} \tag{3.5}$$

so we write $\mathbb{E}\{X(t)^4\} = 24((i)+(ii)+(iii))$, where

$$(i) = \int_{S_t} \mathbb{P}([B_{s_1}+U] = [B_{s_2}+U], [B_{s_3}+U] = [B_{s_4}+U]) ds, \tag{3.6}$$

$$(ii) = \int_{S_t} \mathbb{P}([B_{s_1}+U] = [B_{s_3}+U], [B_{s_2}+U] = [B_{s_4}+U]) ds, \tag{3.7}$$

$$(iii) = \int_{S_t} \mathbb{P}([B_{s_1}+U] = [B_{s_4}+U], [B_{s_2}+U] = [B_{s_3}+U]) ds. \tag{3.8}$$

We first estimate $(ii), (iii)$. Note that $\mathbb{P}([x+U] = [y+U]) = R(x-y)$, by using the estimate $\mathbb{P}(A\bigcap B) \leq \sqrt{\mathbb{P}(A)\mathbb{P}(B)}$, we have

$$(ii) \leq \int_{S_t} \mathbb{E}\{\sqrt{R(B_{s_1}-B_{s_3})}\sqrt{R(B_{s_2}-B_{s_4})}\} ds, \tag{3.9}$$

$$(iii) \leq \int_{S_t} \mathbb{E}\{\sqrt{R(B_{s_1}-B_{s_4})}\sqrt{R(B_{s_2}-B_{s_3})}\} ds. \tag{3.10}$$

Since $R(x) = \prod_{i=1}^d (1-|x_i|)1_{|x_i|<1}$ is bounded and compactly supported, we can find $f \in \mathcal{S}(\mathbb{R}^d)$ such that $\sqrt{R}$ can be replaced by f in the above equalities. Writing by Fourier transform $f(B_{s_i}-B_{s_j}) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{i\xi(B_{s_i}-B_{s_j})}\hat{f}(\xi)d\xi$, we have

$$\frac{1}{t^2}(ii) \leq \frac{t^2}{(2\pi)^{2d}}\int_{\mathbb{R}^{2d}}\int_{u_i\geq 0, \sum_{i=1}^4 u_i \leq 1} e^{-\frac{1}{2}|\xi_1|^2 u_2 t} e^{-\frac{1}{2}|\xi_1+\xi_2|^2 u_3 t} e^{-\frac{1}{2}|\xi_2|^2 u_4 t}\hat{f}(\xi_1)\hat{f}(\xi_2) d\xi_1 d\xi_2 du, \tag{3.11}$$

$$\frac{1}{t^2}(iii) \leq \frac{t^2}{(2\pi)^{2d}}\int_{\mathbb{R}^{2d}}\int_{u_i\geq 0, \sum_{i=1}^4 u_i \leq 1} e^{-\frac{1}{2}|\xi_1|^2 u_2 t} e^{-\frac{1}{2}|\xi_1+\xi_2|^2 u_3 t} e^{-\frac{1}{2}|\xi_1|^2 u_4 t}\hat{f}(\xi_1)\hat{f}(\xi_2) d\xi_1 d\xi_2 du, \tag{3.12}$$

which leads to

$$\frac{1}{t^2}(ii) \leq \frac{4}{(2\pi)^{2d}} \int_{\mathbb{R}^{2d}} \int_{u_i \geq 0, u_1+u_3 \leq 1} e^{-\frac{1}{2}|\xi_1+\xi_2|^2 u_3 t} \frac{1}{|\xi_1|^2|\xi_2|^2} \hat{f}(\xi_1)\hat{f}(\xi_2) d\xi_1 d\xi_2 du_1 du_3, \quad (3.13)$$

$$\frac{1}{t^2}(iii) \leq \frac{4}{(2\pi)^{2d}} \int_{\mathbb{R}^{2d}} \int_{u_i \geq 0, u_1+u_4 \leq 1} e^{-\frac{1}{2}|\xi_1|^2 u_4 t} \frac{1}{|\xi_1|^2|\xi_1+\xi_2|^2} \hat{f}(\xi_1)\hat{f}(\xi_2) d\xi_1 d\xi_2 du_1 du_4. \quad (3.14)$$

By Lebesgue dominated convergence theorem, we see that as $t \to \infty$, $(ii)/t^2 \to 0$ and $(iii)/t^2 \to 0$.

For (i), by change of variables and integrating in U, we have

$$(i) = \sum_k \int_{S_t} \int_{I_0^4} q_{s_2-s_1}(y_2-y_1) q_{s_3-s_2}(y_3-y_2+k) q_{s_4-s_3}(y_4-y_3) dy ds, \quad (3.15)$$

where $I_0 = (-\frac{1}{2}, \frac{1}{2})^d$. We first integrate y_1, y_4, write them in the Fourier domain and denote $\hat{e}(\xi) = \mathcal{F}\{1_{I_0}\} = \prod_{k=1}^d \frac{2\sin(\xi_k)}{\xi_k}$, then change variables $u_i = s_i - s_{i-1}, i = 1,2,4, u_3 = (s_3-s_2)/t$, so

$$\frac{(i)}{t^2} = \frac{1}{t} \sum_k \frac{1}{(2\pi)^{2d}} \int_{u_i \geq 0, u_3 \leq 1, u_1+u_2+u_4 \leq t(1-u_3)} \int_{I_0^2} \int_{\mathbb{R}^{4d}} e^{-\frac{1}{2}|\xi_1|^2 u_2} e^{-\frac{1}{2}|\xi_4|^2 u_4} \hat{e}(\xi_1)\hat{e}(\xi_4) e^{-i\xi_1 y_2} e^{-i\xi_4 y_3} q_{tu_3}(y_3-y_2+k) dy_2 dy_3 d\xi_1 d\xi_4 du. \quad (3.16)$$

Integrating in u_1, u_2, u_4, we have

$$\begin{aligned} &\frac{1}{t} \int_{u_i \geq 0, u_1+u_2+u_4 \leq t(1-u_3)} e^{-\frac{1}{2}|\xi_1|^2 u_2} e^{-\frac{1}{2}|\xi_4|^2 u_4} du_1 du_2 du_4 \\ =&\frac{1-u}{ab}\left(1 + \frac{a}{Tb(b-a)}(1-e^{-bT}) + \frac{b}{Ta(a-b)}(1-e^{-aT})\right) \end{aligned} \quad (3.17)$$

if let $u = u_3, a = \frac{1}{2}|\xi_1|^2, b = \frac{1}{2}|\xi_4|^2, T = t(1-u)$.

We point out that $\frac{a}{Tb(b-a)}(1-e^{-bT}) + \frac{b}{Ta(a-b)}(1-e^{-aT})$ is uniformly bounded by Lemma 5.1. Therefore, $\frac{1}{t}\int_{u_i \geq 0, u_1+u_2+u_4 \leq t(1-u)} e^{-\frac{1}{2}|\xi_1|^2 u_2} e^{-\frac{1}{2}|\xi_4|^2 u_4} du_1 du_2 du_4 = \frac{4(1-u)}{|\xi_1|^2|\xi_4|^2}(1+(*))$ with $(*)$ uniformly bounded and going to zero as $t \to \infty$. Now we have

$$\frac{(i)}{t^2} = \sum_k \frac{1}{(2\pi)^{2d}} \int_{u \in (0,1)} \int_{I_0^2} \int_{\mathbb{R}^{4d}} \frac{4(1-u)}{|\xi_1|^2|\xi_4|^2}(1+(*)) \hat{e}(\xi_1)\hat{e}(\xi_4) e^{-i\xi_1 y_2} e^{-i\xi_4 y_3} q_{tu}(y_3-y_2+k) dy_2 dy_3 d\xi_1 d\xi_4 du. \quad (3.18)$$

Now $\int_{\mathbb{R}^{2d}} \sum_k q_{tu}(y_3-y_2+k) 1_{I_0}(y_2) 1_{I_0}(y_3) e^{-i\xi_1 y_2} e^{-i\xi_4 y_3} dy_2 dy_3$ is uniformly bounded since

$$\begin{aligned} &\left|\int_{\mathbb{R}^{2d}} \sum_k \frac{1}{(2\pi tu)^{d/2}} e^{-|y_2-y_3-k|^2/2tu} 1_{I_0}(y_2) 1_{I_0}(y_3) e^{-i\xi_1 y_2} e^{-i\xi_4 y_3} dy_2 dy_3\right| \\ \leq &\int_{\mathbb{R}^{2d}} \sum_k \frac{1}{(2\pi tu)^{d/2}} e^{-|w-k|^2/2tu} 1_{I_0}(z+w) 1_{I_0}(z) dw dz \leq \sum_k \int_{[-1,1]^d} \frac{1}{(2\pi tu)^{d/2}} e^{-|w-k|^2/2tu} dw = 2^d. \end{aligned} \quad (3.19)$$

In addition, $\frac{1}{|\xi_1|^2|\xi_4|^2}\hat{e}(\xi_1)\hat{e}(\xi_4)$ is integrable, so we have

$$\frac{(i)}{t^2} = \frac{1}{(2\pi)^{2d}} \int_{u \in (0,1)} \int_{I_0^2} \int_{\mathbb{R}^{4d}} \frac{4(1-u)}{|\xi_1|^2|\xi_4|^2}(1+(*)) \hat{e}(\xi_1)\hat{e}(\xi_4) e^{-i\xi_1 y_2} e^{-i\xi_4 y_3} \sum_k q_{tu}(y_3-y_2+k) dy_2 dy_3 d\xi_1 d\xi_4 du. \quad (3.20)$$

On the other hand, for fixed $u \in (0,1)$, we have as $t \to \infty$

$$\begin{aligned}&\int_{\mathbb{R}^{2d}} \sum_k \frac{1}{(2\pi t u)^{d/2}} e^{-|y_3-y_2+k|^2/2tu} 1_{I_0}(y_3) 1_{I_0}(y_2) e^{-i\xi_1 y_2} e^{-i\xi_4 y_3} dy_2 dy_3 \\ \to &\int_{\mathbb{R}^{2d}} 1_{I_0}(y_2) 1_{I_0}(y_3) e^{-i\xi_1 y_2} e^{-i\xi_4 y_3} dy_2 dy_3 = \hat{e}(\xi_1)\hat{e}(\xi_4),\end{aligned} \tag{3.21}$$

and thus by Lebesgue dominated convergence theorem,

$$\frac{(i)}{t^2} \to \frac{1}{(2\pi)^{2d}} \int_0^1 4(1-u)du \int_{\mathbb{R}^{2d}} \frac{1}{|\xi_1|^2|\xi_4|^2} |\hat{e}(\xi_1)|^2 |\hat{e}(\xi_4)|^2 d\xi_1 d\xi_4 = \frac{2}{(2\pi)^{2d}} \left(\int_{\mathbb{R}^d} \frac{\hat{R}(\xi)}{|\xi|^2} d\xi \right)^2. \tag{3.22}$$

To sum up,

$$\mathbb{E}\{(\sum_k p_t(k)^2)^2\} = 8((i)+(ii)+(iii)) \sim \frac{16t^2}{(2\pi)^{2d}} \left(\int_{\mathbb{R}^d} \frac{\hat{R}(\xi)}{|\xi|^2} d\xi \right)^2 = \sigma_d^4 t^2 \tag{3.23}$$

as $t \to \infty$. The proof is complete. □

Remark 3.4. We mention that if $V(x)$ is a mean-zero stationary Gaussian field with fast-decaying covariance function $R \in \mathcal{S}(\mathbb{R}^d)$, and $X(t) = \int_0^t V(B_s)ds$, then $\mathbb{E}\{X(t)^4\} \le Ct^2$ for some uniform constant C. We will need it in the proof of tightness. To see this, we write

$$\frac{1}{t^2}\mathbb{E}\{X(t)^4\} = \frac{24}{t^2} \int_{0\le s_1 \le s_2 \le s_3 \le s_4 \le t} \sum_{\{\tau_i\}=\{s_i\}} \mathbb{E}\{R(B_{\tau_1} - B_{\tau_2}) R(B_{\tau_3} - B_{\tau_4})\} ds, \tag{3.24}$$

and the terms containing $R(B_{s_1} - B_{s_3})R(B_{s_2} - B_{s_4})$ or $R(B_{s_1} - B_{s_4})R(B_{s_2} - B_{s_3})$ go to zero as $t \to \infty$ by the same proof as in Lemma 3.3. For the term containing $R(B_{s_1} - B_{s_2})R(B_{s_3} - B_{s_4})$, we do the same calculation, and get

$$(i) = \frac{1}{t^2} \frac{1}{(2\pi)^{2d}} \int_{0 \le s_1 \le s_2 \le s_3 \le s_4 \le t} \int_{\mathbb{R}^{2d}} e^{-\frac{1}{2}|\xi_1|^2(s_2-s_1)} e^{-\frac{1}{2}|\xi_2|^2(s_4-s_3)} \hat{R}(\xi_1)\hat{R}(\xi_2) d\xi_1 d\xi_2 ds. \tag{3.25}$$

First integrate in s, then let $t \to \infty$, we will see that $(i) \to 2\left(\frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{\hat{R}(\xi)}{|\xi|^2} d\xi \right)^2$. Therefore, we have $\mathbb{E}\{X(t)^4\} \le Ct^2$ for some uniform constant C.

From Lemma 3.2 and 3.3, we know that $\sum_k p_t(k)^2/t \to \sigma_d^2$ in L^2, i.e., the variance of $X_\varepsilon(t)$ conditioning on B, U converges in L^2.

Now we can prove the convergence of finite dimensional distributions, i.e., $\forall 0 = t_0 < t_1 < ... < t_N \le T, \alpha_j \in \mathbb{R}$ fixed, we have

$$\mathbb{E}\{\exp(i \sum_{j=1}^N \alpha_j (X_\varepsilon(t_j) - X_\varepsilon(t_{j-1})))\} \to \exp(-\frac{1}{2}\sigma_d^2 \sum_{j=1}^N \alpha_j^2 (t_j - t_{j-1})) \tag{3.26}$$

as $\varepsilon \to 0$.

Actually, we have

$$\begin{aligned}\sum_{j=1}^{N}\alpha_j(X_\varepsilon(t_j)-X_\varepsilon(t_{j-1})) &= \sum_{j=1}^{N}\alpha_j\varepsilon\sum_k \xi_k(p_{t_j/\varepsilon^2}(k)-p_{t_{j-1}/\varepsilon^2}(k)) \\ &= \sum_k \xi_k\sum_{j=1}^{N}\alpha_j\varepsilon(p_{t_j/\varepsilon^2}(k)-p_{t_{j-1}/\varepsilon^2}(k)).\end{aligned} \tag{3.27}$$

For sum of independent random variables, central limit theorem comes from the following two lemmas by the Lindeberg condition [5].

Lemma 3.5. $\sum_k\left(\sum_{j=1}^N \alpha_j\varepsilon(p_{t_j/\varepsilon^2}(k)-p_{t_{j-1}/\varepsilon^2}(k))\right)^2 \to \sigma_d^2\sum_{j=1}^N\alpha_j^2(t_j-t_{j-1})$ *in probability.*

Proof. We write $\sum_k\left(\sum_{j=1}^N \alpha_j\varepsilon(p_{t_j/\varepsilon^2}(k)-p_{t_{j-1}/\varepsilon^2}(k))\right)^2 = (i)+(ii)$, where

$$(i) \quad = \quad \sum_{j=1}^{N}\alpha_j^2\varepsilon^2\sum_k(p_{t_j/\varepsilon^2}(k)-p_{t_{j-1}/\varepsilon^2}(k))^2, \tag{3.28}$$

$$(ii) \quad = \quad \sum_{i,j=1,\dots N; i\neq j}\alpha_i\alpha_j\varepsilon^2\sum_k(p_{t_i/\varepsilon^2}(k)-p_{t_{i-1}/\varepsilon^2}(k))(p_{t_j/\varepsilon^2}(k)-p_{t_{j-1}/\varepsilon^2}(k)). \tag{3.29}$$

By stationarity and the fact that $(\sum_k p_t(k)^2)^2/t \to \sigma_d^2$ in L^2 we know $(i)\to\sigma_d^2\sum_{j=1}^N\alpha_j^2(t_j-t_{j-1})$ in L^2. So we only have to show $(ii)\to 0$ in probability.

For any $i>j$, consider

$$\mathbb{E}\{\sum_k(p_{t_i/\varepsilon^2}(k)-p_{t_{i-1}/\varepsilon^2}(k))(p_{t_j/\varepsilon^2}(k)-p_{t_{j-1}/\varepsilon^2}(k))\} = \mathbb{E}\{\int_{t_{i-1}/\varepsilon^2}^{t_i/\varepsilon^2}V(B_s)ds\int_{t_{j-1}/\varepsilon^2}^{t_j/\varepsilon^2}V(B_s)ds\}, \tag{3.30}$$

by similar calculation as in the proof of Lemma 3.2, if denote $\delta_1=t_j-t_{j-1},\delta_2=t_{i-1}-t_j,\delta_3=t_i-t_{i-1}$, we have

$$\begin{aligned}&\varepsilon^2\mathbb{E}\{\int_{t_{i-1}/\varepsilon^2}^{t_i/\varepsilon^2}V(B_s)ds\int_{t_{j-1}/\varepsilon^2}^{t_j/\varepsilon^2}V(B_s)ds\} \\ =&\frac{4}{(2\pi)^d}\int_{\mathbb{R}^d}\frac{\varepsilon^2}{|\xi|^4}\left(e^{-\frac12|\xi|^2\frac{\delta_2}{\varepsilon^2}}-e^{-\frac12|\xi|^2\frac{\delta_1+\delta_2}{\varepsilon^2}}-e^{-\frac12|\xi|^2\frac{\delta_2+\delta_3}{\varepsilon^2}}+e^{-\frac12|\xi|^2\frac{\delta_1+\delta_2+\delta_3}{\varepsilon^2}}\right)\hat{R}(\xi)d\xi,\end{aligned} \tag{3.31}$$

and since $\delta_i\geq 0$, by Lebesgue dominated convergence theorem, we have $\mathbb{E}\{\varepsilon^2\sum_k(p_{t_i/\varepsilon^2}(k)-p_{t_{i-1}/\varepsilon^2}(k))(p_{t_j/\varepsilon^2}(k)-p_{t_{j-1}/\varepsilon^2}(k))\}\to 0$, hence $(ii)\to 0$ in probability. The proof is complete. □

Lemma 3.6. $\sup_k \varepsilon p_{t/\varepsilon^2}(k)\to 0$ *in probability.*

Proof. We calculate the $n-$th moment, $\mathbb{E}\{|\sup_k p_{t/\varepsilon^2}(k)|^n\} \leq \sum_k \mathbb{E}\{p_{t/\varepsilon^2}(k)^n\}$. By similar calculation as in the proof of Lemma 3.3, we have

$$\begin{aligned}\sum_k \mathbb{E}\{p_{t/\varepsilon^2}(k)^n\} &= n! \int_{0<s_1<s_2<\ldots<s_n<t/\varepsilon^2} \int_{I_0^n} \prod_{k=2}^n q_{s_k - s_{k-1}}(x_k - x_{k-1}) dx ds \\ &\leq n! \frac{t}{\varepsilon^2} \left(\int_0^{t/\varepsilon^2} \int_{2I_0} q_u(x) dx du \right)^{n-1}\end{aligned} \tag{3.32}$$

with $I_0 = (-\frac{1}{2}, \frac{1}{2})^d$, and we check that $\int_{2I_0} q_u(x)dx < 1_{0<u<1} + \frac{1}{(2\pi u)^{d/2}} 2^d 1_{u\geq 1}$. Because $d \geq 3$, we have $\varepsilon^2 \sum_k \mathbb{E}\{p_{t/\varepsilon^2}(k)^n\}$ is uniformly bounded. So $\mathbb{P}(\sup_k \varepsilon p_{t/\varepsilon^2}(k) > \delta) \leq \frac{C}{\delta^n} \varepsilon^{n-2}$ for some constant C. Pick $n \geq 3$, the proof is complete. □

3.1.2 Tightness

Proposition 3.7. *$X_\varepsilon(t) = \varepsilon \int_0^{t/\varepsilon^2} V(B_s) ds$ is tight in $\mathcal{C}([0,T])$.*

Proof. Consider $X_\varepsilon(t) - X_\varepsilon(s) = \varepsilon \int_{s/\varepsilon^2}^{t/\varepsilon^2} V(B_s) ds$. We claim that $\mathbb{E}\{(X_\varepsilon(t) - X_\varepsilon(s))^4\} \leq C(t-s)^2$ for some uniform constant C. Then tightness follows from [4] given the fact $X_\varepsilon(0) = 0$ and

$$\mathbb{P}(|X_\varepsilon(t) - X_\varepsilon(s)| > \lambda) \leq \frac{C}{\lambda^4}(t-s)^2. \tag{3.33}$$

To prove the claim, we need to bound the fourth moment of $V(x)$. By Lemma 5.2, we have

$$\mathbb{E}\{(X_\varepsilon(t) - X_\varepsilon(s))^4\} \leq C\varepsilon^4 \int_{0\leq s_1 \leq s_2 \leq s_3 \leq s_4 \leq (t-s)/\varepsilon^2} \sum_{\{\tau_i\}=\{s_i\}} \mathbb{E}\{\phi^{\frac{1}{2}}(|B_{\tau_1} - B_{\tau_2}|)\phi^{\frac{1}{2}}(|B_{\tau_3} - B_{\tau_4}|)\} ds \tag{3.34}$$

for some bounded and compactly supported function ϕ. By Remark 3.4, the proof is complete. □

3.2 Convergence to a homogenized equation

Before proving the main theorem, we point out that without much extra effort, Theorem 3.1 can be extended as follows.

Proposition 3.8. *Define $X_\varepsilon(t) = \frac{1}{\varepsilon} \int_0^t \left(V(\frac{B_s^1}{\varepsilon}) - V(\frac{B_s^2}{\varepsilon}) \right) ds$, where B_s^1, B_s^2 are two independent Brownian motions, then*

$$X_\varepsilon(t) \Rightarrow \sqrt{2}\sigma_d W_t \tag{3.35}$$

weakly in $\mathcal{C}([0,T])$, where W_t is a standard Brownian motion.

It is straightforward to check that the proof of tightness does not change since we observe that

$$\mathbb{P}(|X_\varepsilon(t) - X_\varepsilon(s)| > \lambda) \leq \mathbb{P}(|X_{1\varepsilon}(t) - X_{1\varepsilon}(s)| > \frac{\lambda}{2}) + \mathbb{P}(|X_{2\varepsilon}(t) - X_{2\varepsilon}(s)| > \frac{\lambda}{2}) \leq \frac{C}{\lambda^4}(t-s)^2, \tag{3.36}$$

if $X_{i\varepsilon}(t) := \frac{1}{\varepsilon}\int_0^t V(\frac{B_s^i}{\varepsilon})ds$.

For convergence of finite dimensional distributions, since $X_\varepsilon(t)$ has the same distribution as $\varepsilon\sum_k \xi_k(p^1_{t/\varepsilon^2}(k) - p^2_{t/\varepsilon^2}(k))$, where $p_t^i(k) = \int_0^t 1_{[B_s^i+U]=k}ds$, we need to prove Lemma 3.5 and 3.6 with $p_{t/\varepsilon^2}(k)$ replaced by $p^1_{t/\varepsilon^2}(k) - p^2_{t/\varepsilon^2}(k)$. The proof of $\sup_k \varepsilon|p^1_{t/\varepsilon^2}(k) - p^2_{t/\varepsilon^2}| \to 0$ in probability is trivial given Lemma 3.6, so we only need to check the proof of Lemma 3.5.

Let $X_\varepsilon(t) = \varepsilon\int_0^{t/\varepsilon^2}(V(B_s^1) - V(B_s^2))ds$. We look at $\sum_{j=1}^N \alpha_j(X_\varepsilon(t_j) - X_\varepsilon(t_{j-1}))$, where $0 = t_0 < t_1 < ... < t_N \le T$, which could be written as

$$\sum_{j=1}^N \alpha_j(X_\varepsilon(t_j) - X_\varepsilon(t_{j-1})) = \sum_k \xi_k\varepsilon\left(\sum_{j=1}^N \alpha_j(p^1_{t_j/\varepsilon^2}(k) - p^1_{t_{j-1}/\varepsilon^2}(k)) - \sum_{j=1}^N \alpha_j(p^2_{t_j/\varepsilon^2}(k) - p^2_{t_{j-1}/\varepsilon^2}(k))\right). \tag{3.37}$$

Our goal is to show that $\sum_k \varepsilon^2\left(\sum_{j=1}^N \alpha_j(p^1_{t_j/\varepsilon^2}(k) - p^1_{t_{j-1}/\varepsilon^2}(k)) - \sum_{j=1}^N \alpha_j(p^2_{t_j/\varepsilon^2}(k) - p^2_{t_{j-1}/\varepsilon^2}(k))\right)^2$ converges in probability to $2\sigma_d^2\sum_{j=1}^N \alpha_j^2(t_j - t_{j-1})$. We only have to deal with the cross term here, i.e.,

$$\begin{aligned} &|\varepsilon^2\sum_k\sum_{j=1}^N \alpha_j(p^1_{t_j/\varepsilon^2}(k) - p^1_{t_{j-1}/\varepsilon^2}(k))\sum_{j=1}^N \alpha_j(p^2_{t_j/\varepsilon^2}(k) - p^2_{t_{j-1}/\varepsilon^2}(k))| \\ \le&\varepsilon^2\sum_k\sum_{j=1}^N |\alpha_j|(p^1_{t_j/\varepsilon^2}(k) - p^1_{t_{j-1}/\varepsilon^2}(k))\sum_{j=1}^N |\alpha_j|(p^2_{t_j/\varepsilon^2}(k) - p^2_{t_{j-1}/\varepsilon^2}(k)), \end{aligned} \tag{3.38}$$

and the following lemma would conclude the proof of Proposition 3.8.

Lemma 3.9. *$\forall i, j$, we have $\varepsilon^2\mathbb{E}\{\sum_k(p^1_{t_i/\varepsilon^2}(k) - p^1_{t_{i-1}/\varepsilon^2}(k))(p^2_{t_j/\varepsilon^2}(k) - p^2_{t_{j-1}/\varepsilon^2}(k))\} \to 0$ as $\varepsilon \to 0$.*

Proof. We have

$$\begin{aligned} &\varepsilon^2\mathbb{E}\{\sum_k(p^1_{t_i/\varepsilon^2}(k) - p^1_{t_{i-1}/\varepsilon^2}(k))(p^2_{t_j/\varepsilon^2}(k) - p^2_{t_{j-1}/\varepsilon^2}(k))\} = \varepsilon^2\mathbb{E}\{\int_{t_{i-1}/\varepsilon^2}^{t_i/\varepsilon^2} V(B_s^1)ds\int_{t_{j-1}/\varepsilon^2}^{t_j/\varepsilon^2} V(B_s^2)ds\} \\ =&\frac{4}{(2\pi)^d}\int_{\mathbb{R}^d}\varepsilon^2(e^{-\frac{1}{2}|\xi|^2 t_{i-1}/\varepsilon^2} - e^{-\frac{1}{2}|\xi|^2 t_i/\varepsilon^2})(e^{-\frac{1}{2}|\xi|^2 t_{j-1}/\varepsilon^2} - e^{-\frac{1}{2}|\xi|^2 t_j/\varepsilon^2})\frac{1}{|\xi|^4}\hat{R}(\xi)d\xi \to 0 \end{aligned} \tag{3.39}$$

as $\varepsilon \to 0$ by Lebesgue dominated convergence theorem. □

Remark 3.10. Note that the same result holds for $X_\varepsilon(t) := \frac{1}{\varepsilon}\int_0^t\left(V(\frac{B_s^1}{\varepsilon}) + V(\frac{B_s^2}{\varepsilon})\right)ds$. The proof is the same.

Now we are ready to prove our main theorem in the short-range correlation case.

Theorem 3.11. *In the short-range correlation case, we have $\mathbb{E}\{|u_\varepsilon(t,x) - u_0(t,x)|^2\} \to 0$ as $\varepsilon \to 0$, where u_ε and u_0 solve the following PDE, respectively, given the same initial condition $u_\varepsilon(0,x) = u_0(0,x) = f(x)$ for $f \in \mathcal{C}_b(\mathbb{R}^d)$:*

$$\partial_t u_\varepsilon = \frac{1}{2}\Delta u_\varepsilon + i\frac{1}{\varepsilon}V(\frac{x}{\varepsilon})u_\varepsilon, \tag{3.40}$$

$$\partial_t u_0 = \frac{1}{2}\Delta u_0 - \frac{1}{2}\sigma_d^2 u_0. \tag{3.41}$$

Remark 3.12. If we further assume $f \in L^1(\mathbb{R}^d)$, since the solutions are both bounded by Φ, which is in $L^2(\mathbb{R}^d)$ and solves $\Phi_t = \frac{1}{2}\Delta\Phi$ with initial condition $\Phi(0,x) = |f(x)|$, then $\int_{\mathbb{R}^d} \mathbb{E}\{|u_\varepsilon(t,x) - u_0(t,x)|^2\}dx \to 0$ as $\varepsilon \to 0$.

Proof. By the Feynman-Kac formula, we write $u_\varepsilon(t,x) = \mathbb{E}_B\{f(x+B_t)\exp(i\frac{1}{\varepsilon}\int_0^t V(\frac{x+B_s}{\varepsilon})ds)\}$. We claim that $(B_t, \frac{1}{\varepsilon}\int_0^t V(\frac{x+B_s}{\varepsilon})ds) \Rightarrow (N_1, N_2)$ in distribution as $\varepsilon \to 0$ with $N_1 \sim N(0,t), N_2 \sim N(0, \sigma_d^2 t)$ and N_1, N_2 being independent. To see this, since V is stationary, we only have to look at $(\varepsilon B_{t/\varepsilon^2}, \varepsilon\int_0^{t/\varepsilon^2} V(B_s)ds)$ by scaling, and if we compute the characteristic function by conditioning on B, U, we have

$$\begin{aligned}
&\mathbb{E}\{\exp(i\theta_1\varepsilon B_{t/\varepsilon^2} + \theta_2\varepsilon\int_0^{t/\varepsilon^2} V(B_s)ds)\} = \mathbb{E}\{\exp(i\theta_1\varepsilon B_{t/\varepsilon^2})\mathbb{E}\{\exp(i\theta_2\varepsilon\int_0^{t/\varepsilon^2} V(B_s)ds)|B,U\}\} \\
=&\mathbb{E}\{\exp(i\theta_1\varepsilon B_{t/\varepsilon^2})\mathbb{E}\{\exp(i\theta_2\varepsilon\int_0^{t/\varepsilon^2} V(B_s)ds) - \exp(-\frac{1}{2}\sigma_d^2\theta_2^2 t)|B,U\}\} + \mathbb{E}\{\exp(i\theta_1\varepsilon B_{t/\varepsilon^2})\exp(-\frac{1}{2}\sigma_d^2\theta_2^2 t)\} \\
=&(i) + (ii),
\end{aligned} \tag{3.42}$$

and $(ii) = \exp(-\frac{1}{2}\theta_1^2 t - \frac{1}{2}\theta_2^2\sigma_d^2 t)$. Consider

$$\mathbb{E}\{\exp(i\theta_2\varepsilon\int_0^{t/\varepsilon^2} V(B_s)ds) - \exp(-\frac{1}{2}\sigma_d^2\theta_2^2 t)|B,U\} = \mathbb{E}\{\exp(i\theta_2\sum_k \varepsilon\xi_k p_{t/\varepsilon^2}(k)) - \exp(-\frac{1}{2}\sigma_d^2\theta_2^2 t)|B,U\}. \tag{3.43}$$

By Lemma 3.3 and 3.6, for any sequence $\varepsilon_i \to 0$, we can find a subsequence, still denoted by ε_i, such that $\varepsilon_i^2\sum_k p_{t/\varepsilon_i^2}(k)^2 \to \sigma_d^2 t$ and $\sup_k \varepsilon_i p_{t/\varepsilon_i^2}(k) \to 0$ almost surely. So by the central limit theorem, $\sum_k \varepsilon\xi_k p_{t/\varepsilon^2}(k) \Rightarrow N(0, \sigma_d^2 t)$ when freezing B, U. Thus we see $(i) \to 0$, and the claim is proved.

Now we have

$$\begin{aligned}
&\mathbb{E}\{u_\varepsilon(t,x)\} = \mathbb{E}\{f(x+\varepsilon B_{t/\varepsilon^2})\exp(i\varepsilon\int_0^{t/\varepsilon^2} V(B_s)ds)\} \\
\to&\mathbb{E}\{f(x+N_1)\exp(iN_2)\} = \mathbb{E}\{f(x+N_1)\}\mathbb{E}\{\exp(iN_2)\} = \mathbb{E}_B\{f(x+B_t)\}\exp(-\frac{1}{2}\sigma_d^2 t) = u_0(t,x).
\end{aligned} \tag{3.44}$$

Next we consider $|u_\varepsilon(t,x)|^2 = \mathbb{E}_B\{f(x+B_t^1)\overline{f(x+B_t^2)}\exp(i\frac{1}{\varepsilon}\int_0^t(V(\frac{x+B_s^1}{\varepsilon}) - V(\frac{x+B_s^2}{\varepsilon}))ds)\}$ for independent Brownian motions B^1, B^2. By the same discussion as before, we can show

$$(B_t^1, B_t^2, \frac{1}{\varepsilon}\int_0^t(V(\frac{x+B_s^1}{\varepsilon}) - V(\frac{x+B_s^2}{\varepsilon}))ds) \Rightarrow (N_1, N_2, N_3) \tag{3.45}$$

weakly as $\varepsilon \to 0$, where $N_1 \sim N(0,t), N_2 \sim N(0,t), N_3 \sim N(0, 2\sigma_d^2 t)$ and they are independent, so

$$\begin{aligned}
&\mathbb{E}\{|u_\varepsilon(t,x)|^2\} = \mathbb{E}\{f(x+B_t^1)\overline{f(x+B_t^2)}\exp(i\frac{1}{\varepsilon}\int_0^t(V(\frac{x+B_s^1}{\varepsilon}) - V(\frac{x+B_s^2}{\varepsilon}))ds)\} \\
\to&\mathbb{E}\{f(x+N_1)\overline{f(x+N_2)}\exp(iN_3)\} = \mathbb{E}\{f(x+N_1)\}\mathbb{E}\{\overline{f(x+N_2)}\}\mathbb{E}\{\exp(iN_3)\} \\
=&\mathbb{E}_B\{f(x+B_t^1)\}\mathbb{E}_B\{\overline{f(x+B_t^2)}\}\exp(-\sigma_d^2 t) = |u_0(t,x)|^2,
\end{aligned} \tag{3.46}$$

so the proof is complete. □

In the short-range correlation setting, we have assumed that $\{\xi_k\}$ are i.i.d random variables, which simplifies the proof of central limit theorem. However, we expect certain mixing condition would suffice, i.e., Theorem 3.11 should hold when we only assume (2.9) for ϕ satisfying some integrability condition. Note that the order by which different ξ_k join the summation depends on the Brownian path, and when $d \geq 3$, Brownian motion is transient so statistically the mixing condition still holds for the summation of ξ_k, but we need to quantify the transiency of Brownian motion, which is a non-trivial task not considered here.

4 Long-range correlation

We recall that in Theorem 3.11, the constant potential in the limiting equation is given by $\sigma_d = \left(\frac{1}{\pi^{\frac{d}{2}}}\Gamma(\frac{d}{2}-1)\int_{\mathbb{R}^d}\frac{R(x)}{|x|^{d-2}}dx\right)^{\frac{1}{2}}$ so that as long as $R(x)/|x|^{d-2}$ is integrable, σ_d is well-defined. Therefore in the long-range correlation case 1, since $R(x) \sim c_d|x|^{-\alpha}$ with $\alpha > 2$, we still expect a homogenization result to hold.

In the following, we will in turn deal with the cases $\alpha \in (2, d)$ and $\alpha \in (0, 2)$, where the sizes of the potential are chosen to be different, i.e., $\frac{1}{\varepsilon}$ and $\frac{1}{\varepsilon^{\alpha/2}}$. We observe a transition occurring at $\alpha = 2$.

4.1 Case 1: $\alpha \in (2, d)$

In this setting, the proof is similar to the short-range correlation case. Since $\{\xi_k\}$ are assumed to be Gaussian, we do not need to apply the central limit theorem and only have to show the convergence of the variance. We repeat what has been done in Lemma 3.2 and 3.3 with sometimes significantly more involved calculations.

We write $X_\varepsilon(t) = \varepsilon\int_0^{t/\varepsilon^2} V(B_s)ds = \varepsilon\sum_k \xi_k p_{t/\varepsilon^2}(k)$ with the same notation, with a variance conditioning on B, U given by

$$\mathbb{E}\{X_\varepsilon(t)^2|B,U\} = \varepsilon^2\sum_{m,n} r(m-n)p_{t/\varepsilon^2}(m)p_{t/\varepsilon^2}(n), \tag{4.1}$$

and the convergence in probability is given by the following two lemmas.

Lemma 4.1. $\mathbb{E}\{\sum_{m,n} r(m-n)p_t(m)p_t(n)\} \sim \sigma_d^2 t$.

Lemma 4.2. $\mathbb{E}\{(\sum_{m,n} r(m-n)p_t(m)p_t(n))^2\} \sim \sigma_d^4 t^2$.

Proof of Lemma 4.1. Let $X(t) = \int_0^t V(B_s)ds = \sum_k \xi_k p_t(k)$ so that $\mathbb{E}\{\sum_{m,n} r(m-n)p_t(m)p_t(n)\} =$

$\mathbb{E}\{X(t)^2\}$. We have

$$\begin{aligned}\mathbb{E}\{X(t)^2\} &= \int_0^t \int_0^t \mathbb{E}\{R(B_s - B_u)\} ds du = 2 \int_0^t \int_0^s \int_{\mathbb{R}^d} R(x) \left(\frac{1}{2\pi u}\right)^{\frac{d}{2}} \exp(-\frac{|x|^2}{2u}) dx du ds \\ &= \frac{t}{\pi^{d/2}} \int_0^\infty \int_0^\infty \int_{\mathbb{R}^d} \frac{R(x)}{|x|^{d-2}} \lambda^{\frac{d}{2}-2} \exp(-\lambda) 1_{\lambda > \frac{|x|^2}{2st}} 1_{0<s<1} d\lambda ds dx \\ &\sim \frac{t}{\pi^{d/2}} \int_0^\infty \lambda^{\frac{d}{2}-2} \exp(-\lambda) d\lambda \int_{\mathbb{R}^d} \frac{R(x)}{|x|^{d-2}} dx = \frac{t}{\pi^{d/2}} \Gamma(\frac{d}{2}-1) \int_{\mathbb{R}^d} \frac{R(x)}{|x|^{d-2}} dx\end{aligned} \tag{4.2}$$

as $t \to \infty$ by Lebesgue dominated convergence theorem, since $R(x)/|x|^{d-2}$ is integrable. The proof is complete. $\square$

Proof of Lemma 4.2. Since ξ_k are Gaussian, we have $\mathbb{E}\{(\sum_{m,n} r(m-n)p_t(m)p_t(n))^2\} = \frac{1}{3}\mathbb{E}\{X(t)^4\}$ for $X(t) = \int_0^t V(B_s)ds = \sum_k \xi_k p_t(k)$. Given that

$$\begin{aligned}\mathbb{E}\{\prod_{i=1}^4 V(x_i)\} = &\sum_{k_1,k_2} r(k_1)r(k_2)\mathbb{P}([x_1+U]-[x_2+U]=k_1, [x_3+U]-[x_4+U]=k_2) \\ &+ \sum_{k_1,k_2} r(k_1)r(k_2)\mathbb{P}([x_1+U]-[x_3+U]=k_1, [x_2+U]-[x_4+U]=k_2) \\ &+ \sum_{k_1,k_2} r(k_1)r(k_2)\mathbb{P}([x_1+U]-[x_4+U]=k_1, [x_2+U]-[x_3+U]=k_2),\end{aligned} \tag{4.3}$$

we have $\mathbb{E}\{X(t)^4\} = 24((i)+(ii)+(iii))$, where

$$(i) = \sum_{k_1,k_2} r(k_1)r(k_2) \int_{S_t} \mathbb{P}([B_{s_1}+U]-[B_{s_2}+U]=k_1, [B_{s_3}+U]-[B_{s_4}+U]=k_2)ds, \tag{4.4}$$

$$(ii) = \sum_{k_1,k_2} r(k_1)r(k_2) \int_{S_t} \mathbb{P}([B_{s_1}+U]-[B_{s_3}+U]=k_1, [B_{s_2}+U]-[B_{s_4}+U]=k_2)ds, \tag{4.5}$$

$$(iii) = \sum_{k_1,k_2} r(k_1)r(k_2) \int_{S_t} \mathbb{P}([B_{s_1}+U]-[B_{s_4}+U]=k_1, [B_{s_2}+U]-[B_{s_3}+U]=k_2)ds \tag{4.6}$$

for $S_t = \{0 < s_1 < s_2 < s_3 < s_4 < t\}$. Next, we deal with them in turn.

- For (i), after some calculation and change of variables, and let $I_0 = (-\frac{1}{2}, \frac{1}{2})^d$, we have

$$\begin{aligned}\frac{(i)}{t^2} =& \frac{1}{t^2} \sum_{k_1,k_2,k} \int_{S_t} \int_{I_0^4} r(k_1)r(k_2) q_{s_2-s_1}(x_2-x_1+k_1) q_{s_3-s_2}(x_3-x_2+k) q_{s_4-s_3}(x_4-x_3+k_2) dx ds \\ =& \frac{1}{4\pi^d} \int_{\mathbb{R}^{4d}} \int_{\mathbb{R}_+^4} 1_{I_0}(x_1) 1_{I_0}(x_1+y) 1_{I_0}(x_4) 1_{I_0}(x_4-z) \sum_{k_1} \frac{r(k_1)}{|y+k_1|^{d-2}} \sum_{k_2} \frac{r(k_2)}{|z+k_2|^{d-2}} \\ & \sum_k q_{u_3 t}(x_4-z-x_1-y+k) \lambda_2^{\frac{d}{2}-2} e^{-\lambda_2} \lambda_4^{\frac{d}{2}-2} e^{-\lambda_4} 1_{\{u_1+u_3+\frac{|y+k_1|^2}{2\lambda_2 t}+\frac{|z+k_2|^2}{2\lambda_4 t} \le 1\}} dy dz dx_1 dx_4 du_1 du_3 d\lambda_2 d\lambda_4.\end{aligned} \tag{4.7}$$

We note that

$$|1_{I_0}(x_1+y)1_{I_0}(x_4-z)\sum_{k_1}\frac{r(k_1)}{|y+k_1|^{d-2}}\sum_{k_2}\frac{r(k_2)}{|z+k_2|^{d-2}}|\leq\sum_{k_1}1_{2I_0}(y)\frac{|r(k_1)|}{|y+k_1|^{d-2}}\sum_{k_2}1_{2I_0}(z)\frac{|r(k_2)|}{|z+k_2|^{d-2}}, \tag{4.8}$$

and since $\int_{\mathbb{R}^d}\sum_k 1_{2I_0}(y)\frac{|r(k)|}{|y+k|^{d-2}}dy=\int_{\mathbb{R}^d}\frac{1}{|y|^{d-2}}\sum_k 1_{2I_0}(y-k)|r(k)|dy$, by the fact that $|r(k)|\lesssim |k|^{-\alpha}$, we have $\sum_k 1_{2I_0}(y-k)|r(k)|dy\lesssim 1\wedge|y|^{-\alpha}$, thus using the fact that $\alpha>2$ we know $\sum_k 1_{2I_0}(y)\frac{|r(k)|}{|y+k|^{d-2}}$ is integrable. On the other hand, for fixed $u_3\in(0,1)$, we have $\sum_k q_{u_3t}(x_4-z-x_1-y+k)\to 1$ as $t\to\infty$. Moreover, $\int_{\mathbb{R}^{2d}}1_{I_0}(x_1)1_{I_0}(x_4)\sum_k q_{u_3t}(x_4-z-x_1-y+k)dx_1dx_4$ is uniformly bounded. In summary, we can apply Lebesgue dominated convergence theorem to get

$$\begin{aligned}\frac{(i)}{t^2}\to\frac{1}{4\pi^d}\int_{\mathbb{R}^{4d}}\int_{\mathbb{R}_+^4}1_{I_0}(x_1)1_{I_0}(x_1+y)1_{I_0}(x_4)1_{I_0}(x_4-z)\sum_{k_1}\frac{r(k_1)}{|y+k_1|^{d-2}}\sum_{k_2}\frac{r(k_2)}{|z+k_2|^{d-2}}\\ \lambda_2^{\frac{d}{2}-2}e^{-\lambda_2}\lambda_4^{\frac{d}{2}-2}e^{-\lambda_4}1_{\{u_1+u_3\leq 1\}}dydzdx_1dx_4du_1du_3d\lambda_2d\lambda_4.\end{aligned} \tag{4.9}$$

We check that

$$\begin{aligned}RHS=&\frac{1}{8\pi^d}\Gamma(\frac{d}{2}-1)^2\left(\int_{\mathbb{R}^{2d}}1_{I_0}(x)1_{I_0}(x+y)\sum\frac{r(k)}{|y+k|^{d-2}}dxdy\right)^2\\ =&\frac{1}{8\pi^d}\Gamma(\frac{d}{2}-1)^2\left(\int_{\mathbb{R}^d}\frac{R(x)}{|x|^{d-2}}dx\right)^2.\end{aligned} \tag{4.10}$$

- (ii). By a similar calculation, we have

$$\begin{aligned}\frac{|(ii)|}{t^2}=&|\frac{1}{t^2}\sum_{k_1,k_2,k}\int_{S_t}\int_{I_0^4}r(k_1)r(k_2)q_{s_2-s_1}(x_2-x_1+k+k_1)q_{s_3-s_2}(x_3-x_2-k)q_{s_4-s_3}(x_4-x_3+k+k_2)dxds|\\ \lesssim&\sum_{k_1,k_2,k}\int_{\mathbb{R}^{3d}}\int_{\mathbb{R}_+^2}1_{2I_0}(x)1_{2I_0}(y)1_{2I_0}(z)\frac{|r(k_1-k)|}{|y+k_1|^{d-2}}\frac{|r(k_2-k)|}{|z+k_2|^{d-2}}q_{tu_3}(x-k)1_{u_1+u_3\leq 1}dxdydzdu_1du_3.\end{aligned} \tag{4.11}$$

Define $G(y-k)=\sum_n 1_{2I_0}(y-k-n)|r(n)|$ and $F(k)=\int_{\mathbb{R}^d}\frac{1}{|y|^{d-2}}G(y-k)dy$. It is straightforward to check that $G(y-k)\lesssim 1\wedge|y-k|^{-\alpha}$, and thus by Lemma 5.3, we have $F(k)\lesssim 1\wedge|k|^{-\alpha+2}$, which leads to

$$\begin{aligned}\frac{|(ii)|}{t^2}\lesssim&\sum_k\int_{\mathbb{R}^d}\int_{\mathbb{R}_+^2}1_{2I_0}(x)(1\wedge\frac{1}{|k|^{\alpha-2}})^2q_{tu_3}(x-k)1_{u_1+u_3\leq 1}dxdu_1du_3\\ =&\int_{\mathbb{R}_+^2}\int_{\mathbb{R}^d}q_{tu_3}(x)\sum_k 1_{2I_0}(x-k)(1\wedge\frac{1}{|k|^{2\alpha-4}})dx1_{u_1+u_3\leq 1}du_1du_3.\end{aligned} \tag{4.12}$$

Since $\sum_k 1_{2I_0}(x-k)(1\wedge\frac{1}{|k|^{2\alpha-4}})\lesssim 1\wedge|x|^{-2\alpha+4}$, by Lemma 5.4 and Lebesgue dominated convergence theorem, $\frac{|(ii)|}{t^2}\to 0$ as $t\to\infty$.

- (iii). Similarly, we have

$$\begin{aligned}\frac{(iii)}{t^2} =& \frac{1}{t^2}\sum_{k_1,k_2,k}\int_{I_0^4} r(k_1)r(k_2)q_{s_2-s_1}(x_2-x_1+k+k_1)q_{s_3-s_2}(x_3-x_2-k_2)q_{s_4-s_3}(x_4-x_3-k+k_2)dxds\\ =& \frac{1}{t^2}\sum_{k_1,k_2,k}\int_{I_0^4} r(k_1-k_2-k)r(k_2)q_{s_2-s_1}(x_2-x_1+k_1)q_{s_3-s_2}(x_3-x_2-k_2)q_{s_4-s_3}(x_4-x_3-k)dxds,\end{aligned} \tag{4.13}$$

and with a change of variables,

$$\frac{|(iii)|}{t^2} \lesssim \sum_{k_1,k_2,k}\int_{\mathbb{R}^{3d}}\int_{\mathbb{R}_+^2} 1_{2I_0}(x)1_{2I_0}(y)1_{2I_0}(z)\frac{|r(k_1-k_2-k)|}{|x+k_1|^{d-2}}\frac{|r(k_2)|}{|y-k_2|^{d-2}}q_{tu_3}(z-k)1_{u_1+u_3\le 1}dxdydzdu_1du_3. \tag{4.14}$$

First of all, by the same discussion as for (ii), we have

$$\int_{\mathbb{R}^d}\sum_{k_1} 1_{2I_0}(x)\frac{|r(k_1-k_2-k)|}{|x+k_1|^{d-2}}dx \lesssim 1\wedge|k_2+k|^{2-\alpha}. \tag{4.15}$$

Next, we need to estimate $(*) = \int_{\mathbb{R}^d}\sum_{k_2} 1_{2I_0}(y)\frac{|r(k_2)|}{|y-k_2|^{d-2}}(1\wedge\frac{1}{|k_2+k|^{\alpha-2}})dy$, and we write $(*) = \int_{\mathbb{R}^d}\frac{1}{|y|^{d-2}}F(y,k)dy$ for

$$F(y,k) = \sum_{k_2} 1_{2I_0}(y-k_2)(1\wedge\frac{1}{|k_2|^{\alpha}})(1\wedge\frac{1}{|k-k_2|^{\alpha-2}}). \tag{4.16}$$

Since $F(y,k)\lesssim 1\wedge|y|^{-\alpha}$, $(*)$ is uniformly bounded. Now we consider the case when $|k|$ is sufficiently large, and write $(*) = (1)+(2)$ with

$$(1) \lesssim \int_{|y|<M}\frac{1}{|y|^{d-2}}\sum_{k_2} 1_{2I_0}(y-k_2)\frac{1}{|k-k_2|^{\alpha-2}}dy, \tag{4.17}$$

$$(2) \lesssim \int_{|y|\ge M}\frac{1}{|y|^{d-2}}\sum_{k_2} 1_{2I_0}(y-k_2)\frac{1}{|k_2|^{\alpha}}(1\wedge\frac{1}{|k-k_2|^{\alpha-2}})dy \tag{4.18}$$

for some fixed constant M. Note that for (1) we have $|k_2|\le M+\sqrt{d}$, so $(1)\lesssim\frac{1}{|k|^{\alpha-2}}$. For (2), we have $\sum_{k_2} 1_{2I_0}(y-k_2)\frac{1}{|k_2|^{\alpha}}(1\wedge\frac{1}{|k-k_2|^{\alpha-2}})\lesssim\sum_{k_2} 1_{2I_0}(y-k_2)\frac{1}{|y|^{\alpha}}(1\wedge\frac{1}{|k-k_2|^{\alpha-2}})$, and thus

$$(2)\lesssim\int_{|y|\ge M}\frac{1}{|y|^{d-2+\alpha}}\sum_{k_2} 1_{2I_0}(y-k-k_2)(1\wedge\frac{1}{|k_2|^{\alpha-2}})dy. \tag{4.19}$$

Let $G(y-k) = \sum_{k_2} 1_{2I_0}(y-k-k_2)(1\wedge\frac{1}{|k_2|^{\alpha-2}})\lesssim 1\wedge|y-k|^{2-\alpha}$. Therefore,

$$(2)\lesssim\int_{|y|\ge M}\frac{1}{|y|^{d-2+\alpha}}(1\wedge\frac{1}{|k-y|^{\alpha-2}})dy. \tag{4.20}$$

Note that the RHS does not fall into the form of integral in Lemma 5.3, but following the same prove as in the lemma, we can show $(2)\lesssim\frac{1}{|k|^{\alpha-2}}$. To sum up, we have shown that $(*)\lesssim 1\wedge\frac{1}{|k|^{\alpha-2}}$, so $\int_{\mathbb{R}^d}\int_{\mathbb{R}_+^2}\sum_k 1_{2I_0}(z)(*)q_{tu_3}(z-k)1_{u_1+u_3\le 1}dzdu_1du_3\to 0$ as $t\to\infty$.

The proof is complete. $\square$

From Lemma 4.1 and 4.2, we have

$$\mathbb{E}\{X_\varepsilon(t)^2|B,U\} = \varepsilon^2 \sum_{m,n} r(m-n)p_{t/\varepsilon^2}(m)p_{t/\varepsilon^2}(n) \to \sigma_d^2 t \tag{4.21}$$

in L^2 as $\varepsilon \to 0$. We claim that $X_\varepsilon(t)$ converges weakly to some Gaussian $N(0,\sigma_d^2 t)$. To see this, we only have to note that

$$\mathbb{E}\{e^{i\theta X_\varepsilon(t)}\} = \mathbb{E}\{\mathbb{E}\{e^{i\theta\varepsilon\sum_k \xi_k p_{t/\varepsilon^2}(k)}|B,U\}\} = \mathbb{E}\{e^{-\frac{1}{2}\theta^2\varepsilon^2\sum_{m,n} r(m-n)p_{t/\varepsilon^2}(m)p_{t/\varepsilon^2}(n)}\}. \tag{4.22}$$

Before proving the main theorem, we show the following lemma, which basically says that the interaction between two independent Brownian motions in a random scenery is small.

Lemma 4.3. *For any function $R(x)$ such that $R(x)/|x|^{d-2}$ is integrable, we have*

$$\frac{1}{t}\int_0^t\int_0^t R(B^1(s)-B^2(u))dsdu \to 0 \tag{4.23}$$

in probability as $t\to\infty$, where B^1,B^2 are two independent Brownian motions.

Proof. In the proof we could assume $R(x)\geq 0$ and by direct calculation, we have

$$\begin{aligned}
&\frac{1}{t}\int_0^t\int_0^t \mathbb{E}\{R(B^1(s)-B^2(u))\}dsdu = \frac{1}{t}\int_0^t\int_0^t\int_{\mathbb{R}^d} R(x)q_{s+u}(x)dxdsdu\\
=&\frac{1}{t}\int_{R_+^2}\int_{\mathbb{R}^d}\frac{1}{2\pi^{\frac{d}{2}}}\frac{R(x)}{|x|^{d-2}}\lambda^{\frac{d}{2}-2}\exp(-\lambda)1_{(0,t)}(u)1_{(\frac{|x|^2}{2(t+u)},\frac{|x|^2}{2u})}(\lambda)d\lambda dxdu\\
=&\int_{R_+^2}\int_{\mathbb{R}^d}\frac{1}{2\pi^{\frac{d}{2}}}\frac{R(x)}{|x|^{d-2}}\lambda^{\frac{d}{2}-2}\exp(-\lambda)1_{(0,1)}(u)1_{(\frac{|x|^2}{2t(1+u)},\frac{|x|^2}{2tu})}(\lambda)d\lambda dxdu,
\end{aligned} \tag{4.24}$$

and now we only need the Lebesgue dominated convergence theorem to finish the proof. $\square$

Now we can prove the main theorem for this case.

Theorem 4.4. *In the long-range correlation case 1, we have $\mathbb{E}\{|u_\varepsilon(t,x)-u_0(t,x)|^2\}\to 0$ as $\varepsilon\to 0$, where u_ε, u_0 solves the following PDE respectively given the same initial condition $u_\varepsilon(0,x) = u_0(0,x) = f(x)$ for $f\in\mathcal{C}_b(\mathbb{R}^d)$:*

$$\partial_t u_\varepsilon = \frac{1}{2}\Delta u_\varepsilon + i\frac{1}{\varepsilon}V(\frac{x}{\varepsilon})u_\varepsilon, \tag{4.25}$$

$$\partial_t u_0 = \frac{1}{2}\Delta u_0 - \frac{1}{2}\sigma_d^2 u_0. \tag{4.26}$$

Remark 4.5. Similar to the short-range correlation case, we have $\int_{\mathbb{R}^d}\mathbb{E}\{|u_\varepsilon(t,x)-u_0(t,x)|^2\}dx\to 0$ as $\varepsilon\to 0$ if $f\in L^1(\mathbb{R}^d)$.

Proof. The proof is similar with what has been done for Theorem 3.11, so we do not present all the details here.

First of all, we show the convergence of $\mathbb{E}\{u_\varepsilon(t,x)\}$. Since $X_\varepsilon(t) = \varepsilon \int_0^{t/\varepsilon^2} V(B_s)ds$ weakly converges to some Gaussian $N(0, \sigma_d^2 t)$, we only have to show that $\varepsilon B_{t/\varepsilon^2}$ and $X_\varepsilon(t)$ are asymptotically independent, and this is the same as in Theorem 3.11. Therefore, we have $\mathbb{E}\{u_\varepsilon(t,x)\} \to \mathbb{E}_B\{f(x+B_t)\}\exp(-\frac{1}{2}\sigma_d^2 t) = u_0(t,x)$.

Secondly, we prove the convergence of

$$\mathbb{E}\{|u_\varepsilon(t,x)|^2\} = \mathbb{E}\{f(x+\varepsilon B^1_{t/\varepsilon^2})\overline{f(x+\varepsilon B^2_{t/\varepsilon^2})}\exp(i\varepsilon\int_0^{t/\varepsilon^2}(V(B^1_s)-V(B^2_s))ds)\} \tag{4.27}$$

for independent Brownian motions B^1, B^2. We point out that $\varepsilon\int_0^{t/\varepsilon^2}(V(B^1_s)-V(B^2_s))ds$ converges weakly to some Gaussian $N(0, 2\sigma_d^2 t)$, which comes from the fact that ξ_k are Gaussian and Lemma 4.1, 4.2, 4.3. The rest of the proof is the same as in Theorem 3.11.

The proof is complete. $\square$

4.2 Case 2: $\alpha \in (0,2)$

In this setting, we first present some properties of the random field. Since Φ satisfies $\mathbb{E}\{\Phi(\eta_k)^2\} < \infty$, we can write $\xi_k = \Phi(\eta_k)$ in Hermite polynomial expansion

$$\xi_k = \Phi(\eta_k) = \sum_{n=0}^{\infty} \frac{V_n}{n!} H_n(\eta_k), \tag{4.28}$$

where $H_n(x) = (-1)^n \exp(\frac{x^2}{2})\frac{d^n}{dx^n}\exp(-\frac{x^2}{2})$ is the $n-$th order Hermite polynomial and $V_n = \mathbb{E}\{H_n(\eta_k)\Phi(\eta_k)\}$. By the assumption that $V_0 = 0, V_1 \neq 0$, we have

$$\begin{aligned} r(k) = \mathbb{E}\{\xi_0\xi_k\} =& \mathbb{E}\{\Phi(\eta_0)\Phi(\eta_k)\} = \sum_{n=0}^{\infty}\frac{V_n^2}{(n!)^2}\mathbb{E}\{H_n(\eta_0)H_n(\eta_k)\} \\ =& \sum_{n=0}^{\infty}\frac{V_n^2}{n!}\rho(k)^n = V_1^2\rho(k) + \sum_{n=2}^{\infty}\frac{V_n^2}{n!}\rho(k)^n. \end{aligned} \tag{4.29}$$

Since $\sum_{n=0}^{\infty}\frac{V_n^2}{n!} < \infty$, we see that $r(k) \sim V_1^2\rho(k)$ as $|k| \to \infty$. By (2.6) and the fact that $\rho(k) \sim c_d \prod_{i=1}^d |k_i|^{-\alpha_i}$, we know that $R(x) \sim V_1^2 c_d \prod_{i=1}^d |x_i|^{-\alpha_i}$ as $\min_{i=1,\ldots,d}|x_i| \to \infty$. Because $\alpha = \sum_{i=1}^d \alpha_i < 2$, the correlation function $R(x)$ decays sufficiently slowly so that the integral $X_\varepsilon(t) = \frac{1}{\varepsilon^{\alpha/2}}\int_0^t V(\frac{B_s}{\varepsilon})ds$ is able to memorize the Brownian path in the limit $\varepsilon \to 0$.

The assumption that $V_1 \neq 0$ is crucial for the appearance of Gaussian white noise in the limiting equation, and it turns out that we can reduce it to the case $\Phi(x) = x$, namely $V(x) = \eta_{[x+U]}$, so conditioning on B, U, $X_\varepsilon(t) = \frac{1}{\varepsilon^{\alpha/2}}\int_0^t V(\frac{B_s}{\varepsilon})ds$ is Gaussian, and we can prove its weak convergence by calculating the variance. Before doing that, following [10] we define the solution to the limiting SPDE.

4.2.1 limiting SPDE

We first define the formally-written random variable $\int_0^t \int_{\mathbb{R}^d} \delta(x - B_s)W(dx)ds$, where $W(dx)$ is the generalized Gaussian random field independent from Brownian motion B_t, and the covariance function $\mathbb{E}\{W(dx)W(dy)\} = \frac{1}{\prod_{i=1}^d |x_i - y_i|^{\alpha_i}} dxdy$.

Proposition 4.6. *Assume $\sum_{i=1}^d \alpha_i < 2$ and define $Y_\varepsilon(t) = \int_0^t \int_{\mathbb{R}^d} q_\varepsilon(x - B_s)W(dx)ds$. Then $Y_\varepsilon(t)$ converges in L^2 as $\varepsilon \to 0$ to some random variable $Y(t)$, denoted as $Y(t) = \int_0^t \int_{\mathbb{R}^d} \delta(x - B_s)W(dx)ds$. When conditioning on B, then Y_t is a Gaussian random variable with zero mean and variance*

$$\mathbb{E}\{Y(t)^2|B\} = \int_0^t \int_0^t \frac{1}{\prod_{i=1}^d |B_i(s) - B_i(u)|^{\alpha_i}} dsdu, \tag{4.30}$$

where $B_i(s)$ denotes the $i-th$ component of B_s.

Proof. We first point out that the RHS of (4.30) is almost surely finite, and this comes from the fact that $\sum_{i=1}^d \alpha_i < 2$ and

$$\mathbb{E}\{\int_0^t \int_0^t \frac{1}{\prod_{i=1}^d |B_i(s) - B_i(u)|^{\alpha_i}} dsdu\} = \int_0^t \int_0^t \frac{1}{|s-u|^{\sum_{i=1}^d \alpha_i/2}} dsdu \prod_{i=1}^d \mathbb{E}\{|N|^{-\alpha_i}\}, \tag{4.31}$$

where $N \sim N(0,1)$. Secondly, we calculate

$$\mathbb{E}\{Y_\varepsilon^2(t)\} = \int_0^t \int_0^t \int_{\mathbb{R}^{2d}} \mathbb{E}\{q_\varepsilon(x - B_s)q_\varepsilon(y - B_u)\} \frac{1}{\prod_{i=1}^d |x_i - y_i|^{\alpha_i}} dxdydsdu. \tag{4.32}$$

By Lemma 5.5, $\int_{\mathbb{R}^{2d}} q_\varepsilon(x - B_s)q_\varepsilon(y - B_u)\frac{1}{\prod_{i=1}^d |x_i - y_i|^{\alpha_i}} dxdy \to \frac{1}{\prod_{i=1}^d |B_i(s) - B_i(u)|^{\alpha_i}}$ as $\varepsilon \to 0$. By Lemma 5.6, i.e., the inequality $\int_{\mathbb{R}^2} q_\varepsilon(x_1 + y_1)q_\varepsilon(x_2 + y_2)|y_1 - y_2|^{-\alpha} dy_1 dy_2 \le C|x_1 - x_2|^{-\alpha}$ as long as $\alpha \in (0,1)$, and by the Lebesgue dominated convergence theorem, we have the convergence $\mathbb{E}\{Y_\varepsilon^2(t)\} \to \int_0^t \int_0^t \mathbb{E}\{\frac{1}{\prod_{i=1}^d |B_i(s) - B_i(u)|^{\alpha_i}}\} dsdu$. Similarly, we can show $\mathbb{E}\{Y_{\varepsilon_1}(t)Y_{\varepsilon_2}(t)\} \to \int_0^t \int_0^t \mathbb{E}\{\frac{1}{\prod_{i=1}^d |B_i(s) - B_i(u)|^{\alpha_i}}\} dsdu$ as $\varepsilon_1, \varepsilon_2 \to 0$. Thus, we have shown that $\{Y_\varepsilon(t)\}$ is a Cauchy sequence in L^2 since $\lim_{\varepsilon_1, \varepsilon_2 \to 0} \mathbb{E}\{(Y_{\varepsilon_1}(t) - Y_{\varepsilon_2}(t))^2\} = 0$. The limit is then denoted as $Y(t) = \int_0^t \int_{\mathbb{R}^d} \delta(x - B_s)W(dx)ds$.

Next, we consider the conditional distribution. Since $Y_\varepsilon(t) \to Y(t)$ in L^2, there exists a subsequence ε_k such that $Y_{\varepsilon_k}(t) \to Y(t)$ almost surely. Note that $W(dx)$ and B_t are independent, so the probability space is the product space. Then we know that conditioning on the Brownian motion, $Y_{\varepsilon_k}(t) \to Y(t)$ almost surely as $k \to \infty$, and this leads to convergence in distribution. Given B, $Y_\varepsilon(t)$ is Gaussian with variance

$$\begin{aligned}\mathbb{E}\{Y_\varepsilon^2(t)|B\} &= \int_0^t \int_0^t \int_{\mathbb{R}^{2d}} q_\varepsilon(x - B_s)q_\varepsilon(y - B_u)\frac{1}{\prod_{i=1}^d |x_i - y_i|^{\alpha_i}} dxdydsdu \\ &\to \int_0^t \int_0^t \frac{1}{\prod_{i=1}^d |B_i(s) - B_i(u)|^{\alpha_i}} dsdu.\end{aligned} \tag{4.33}$$

The proof is complete. □

Remark 4.7. If we define $Y^i(t) = \int_0^t \int_{\mathbb{R}^d} \delta(x - B_s^i) W(dx) ds$ for independent Brownian motions B^1, B^2, the same proof implies that $Y^1(t), Y^2(t)$ are jointly Gaussian with covariance function given by $\mathbb{E}\{Y^1(t)Y^2(t)|B\} = \int_0^t \int_0^t \frac{1}{\prod_{i=1}^d |B_i^1(s) - B_i^2(u)|^{\alpha_i}} dsdu$ when conditioning on B^1, B^2.

Remark 4.8. From the proof of Proposition 4.6, we see that the distribution of $\int_0^t \int_{\mathbb{R}^d} \delta(x - B_s) W(dx) ds$ does not depend on the starting point of Brownian motion.

With random variable $\int_0^t \int_{\mathbb{R}^d} \delta(x - B_s) W(dx) ds$, we can formally write the solution to the SPDE

$$\partial_t u = \frac{1}{2} \Delta u + i \dot{W} u \tag{4.34}$$

with initial condition $u(0, x) = f(x)$ by Feynman-Kac formula as

$$u(t, x) = \mathbb{E}_B\{f(x + B_t) \exp(i \int_0^t \int_{\mathbb{R}^d} \delta(y - x - B_s) W(dy) ds)\}. \tag{4.35}$$

We point that the $u(t, x)$ defined as above coincides with the usual definition of weak solution to SPDE (4.34):

Definition 4.9. *A random field $u(t, x)$ is a weak solution to (4.34) if for any $\mathcal{C}^\infty$ function ϕ with compact support we have*

$$\int_{\mathbb{R}^d} u(t, x)\phi(x) dx = \int_{\mathbb{R}^d} f(x)\phi(x) dx + \frac{1}{2} \int_0^t \int_{\mathbb{R}^d} u(s, x) \Delta\phi(x) dx ds + i \int_0^t \int_{\mathbb{R}^d} u(s, x)\phi(x) W(dx) ds. \tag{4.36}$$

Proposition 4.10. *If $\sum_{i=1}^d \alpha_i < 2$, $u(t, x)$ is the weak solution to (4.34).*

The proof is a direct adaption of Theorem 4.3 in [10], and we do not present it here.

4.2.2 Convergence to a stochastic equation

First of all we write $X_\varepsilon(t) = \frac{1}{\varepsilon^{\alpha/2}} \int_0^t V(\frac{B_s}{\varepsilon}) ds = \frac{1}{\varepsilon^{\alpha/2}} \sum_k \Phi(\eta_k) p_t(k)$, where we have defined $p_t(k) = \int_0^t 1_{[\frac{B_s}{\varepsilon} + U] = k} ds = \int_0^t 1_{B_s \in \varepsilon(I_k - U)} ds$ and $I_k = \times_{i=1}^d [k_i, k_i + 1)$. We reduce it to the Gaussian case by the following lemma:

Lemma 4.11. $Y_\varepsilon(t) := \frac{1}{\varepsilon^{\alpha/2}} \sum_k (\Phi(\eta_k) - V_1 \eta_k) p_t(k) \to 0$ *in probability as* $\varepsilon \to 0$.

Proof. Conditioning on B, U, we have

$$\begin{aligned} \mathbb{E}\{Y_\varepsilon(t)^2 | B, U\} =& \frac{1}{\varepsilon^\alpha} \sum_{m,n} \sum_{k=2}^\infty \frac{V_k^2}{n!} \rho(m - n)^k p_t(m) p_t(n) \\ \leq& \frac{C}{\varepsilon^\alpha} \sum_{m,n} \rho(m - n)^2 p_t(m) p_t(n) \end{aligned} \tag{4.37}$$

for some constant C, and the RHS can be written as

$$RHS = \frac{C}{\varepsilon^\alpha} \int_0^t \int_0^t \rho_\varepsilon(\frac{B_s - B_u}{\varepsilon})^2 ds du \tag{4.38}$$

if we define $\rho_\varepsilon(\frac{1}{\varepsilon}(x-y)) = \rho(\frac{1}{\varepsilon}\varepsilon(m-n))$ if $x \in \varepsilon(I_m - U), y \in \varepsilon(I_n - U)$. Since ρ is bounded, and by Lemma 4.13, we have

$$\begin{aligned} RHS \leq & C \sup_{|k|\geq M-\sqrt{d}} |\rho(k)| \int_0^t \int_0^t \frac{1}{\prod_{i=1}^d |B_i(s) - B_i(u)|^{\alpha_i}} 1_{|B_s - B_u| > M\varepsilon} dsdu \\ & + \frac{C}{\varepsilon^\alpha} \int_0^t \int_0^t 1_{|B_s - B_u| \leq M\varepsilon} dsdu, \end{aligned} \tag{4.39}$$

which leads to

$$\mathbb{E}\{Y_\varepsilon(t)^2\} \leq C \sup_{|k|\geq M-\sqrt{d}} |\rho(k)| + \frac{C}{\varepsilon^\alpha} \int_0^t \int_0^t \mathbb{P}(|B_s - B_u| \leq M\varepsilon) dsdu. \tag{4.40}$$

By Lemma 5.7, first let $\varepsilon \to 0$, then $M \to \infty$, the proof is complete. $\square$

Now we can prove the weak convergence of $X_\varepsilon(t)$.

Proposition 4.12. *$X_\varepsilon(t) \Rightarrow V_1\sqrt{c_d} \int_0^t \int_{\mathbb{R}^d} \delta(x - B_s) W(dx) ds$ as $\varepsilon \to 0$.*

Proof. We write $X_\varepsilon(t) = \frac{1}{\varepsilon^{\alpha/2}} \sum_k (\Phi(\eta_k) - V_1\eta_k) p_t(k) + \frac{1}{\varepsilon^{\alpha/2}} \sum_k V_1 \eta_k p_t(k)$, and by Lemma 4.11, we only need to show the weak convergence of $\frac{1}{\varepsilon^{\alpha/2}} \sum_k V_1 \eta_k p_t(k)$.

By conditioning on B, U, we calculate the characteristic function

$$\mathbb{E}\{\exp(i\theta \frac{1}{\varepsilon^{\alpha/2}} \sum_k V_1 \eta_k p_t(k))\} = \mathbb{E}\{\exp(-\frac{V_1^2\theta^2}{2\varepsilon^\alpha} \sum_{m,n} \rho(m-n) p_t(m) p_t(n))\}, \tag{4.41}$$

and claim that $\frac{1}{\varepsilon^\alpha} \sum_{m,n} \rho(m-n) p_t(m) p_t(n) \to c_d \int_0^t \int_0^t \frac{1}{\prod_{i=1}^d |B_i(s) - B_i(u)|^{\alpha_i}} dsdu$ almost surely.

Actually, we write

$$\frac{1}{\varepsilon^\alpha} \sum_{m,n} \rho(\frac{1}{\varepsilon}\varepsilon(m-n)) p_t(m) p_t(n) = \frac{1}{\varepsilon^\alpha} \int_0^t \int_0^t \rho_\varepsilon(\frac{1}{\varepsilon}(B_s - B_u)) dsdu \tag{4.42}$$

where we have also defined $\rho_\varepsilon(\frac{1}{\varepsilon}(x-y)) = \rho(\frac{1}{\varepsilon}\varepsilon(m-n))$ if $x \in \varepsilon(I_m - U), y \in \varepsilon(I_n - U)$. Thus by our assumptions on $\rho(k)$, we see that $\rho_\varepsilon(\frac{1}{\varepsilon}(x-y)) \sim c_d \varepsilon^\alpha \prod_{i=1}^d |x_i - y_i|^{-\alpha_i}$ for all x, y satisfying $\forall i, x_i \neq y_i$. Now we only need a bound to pass to the limit, and this is given by Lemma 4.13. Therefore, we have

$$\begin{aligned} \mathbb{E}\{\exp(i\theta \frac{1}{\varepsilon^{\alpha/2}} \sum_k V_1 \eta_k p_t(k))\} \to & \mathbb{E}\{\exp(-\frac{1}{2}\theta^2 V_1^2 c_d \int_0^t \int_0^t \frac{1}{\prod_{i=1}^d |B_i(s) - B_i(u)|^{\alpha_i}} dsdu)\} \\ = & \mathbb{E}\{\exp(i\theta V_1 \sqrt{c_d} \int_0^t \int_{\mathbb{R}^d} \delta(x - B_s) W(dx) ds)\} \end{aligned} \tag{4.43}$$

as $\varepsilon \to 0$. $\square$

Lemma 4.13. *$|\rho_\varepsilon(\frac{1}{\varepsilon}(x-y))| \leq C\varepsilon^\alpha \prod_{i=1}^d |x_i - y_i|^{-\alpha_i}$ for some uniform constant C independent of ε and x, y.*

Proof. Pick $M > 0, x, y$ fixed. Let $A = \{i : |x_i - y_i| \le M\varepsilon\}$, since $\rho_\varepsilon(\frac{1}{\varepsilon}(x-y)) = \rho(\frac{1}{\varepsilon}\varepsilon(m-n))$, if $i \notin A$, i.e., $|x_i - y_i| > M\varepsilon$, we have $|m_i - n_i| > M - 1$. So

$$|\rho_\varepsilon(\frac{1}{\varepsilon}(x-y))| \le C\prod_{i\notin A}\frac{\varepsilon^{\alpha_i}}{\varepsilon^{\alpha_i}|m_i-n_i|^{\alpha_i}} \le C\prod_{i\notin A}\frac{\varepsilon^{\alpha_i}}{||x_i-y_i|-\varepsilon|^{\alpha_i}} = C\prod_{i\notin A}\varepsilon^{\alpha_i}\frac{|x_i-y_i|^{\alpha_i}}{||x_i-y_i|-\varepsilon|^{\alpha_i}}\frac{1}{|x_i-y_i|^{\alpha_i}}$$
$$\le C\prod_{i\notin A}\varepsilon^{\alpha_i}\frac{1}{(1-1/M)^{\alpha_i}}\frac{1}{|x_i-y_i|^{\alpha_i}}. \tag{4.44}$$

On the other hand, for $i \in A$, we have $\prod_{i\in A}\varepsilon^{\alpha_i}|x_i-y_i|^{-\alpha_i} \ge \prod_{i\in A} M^{-\alpha_i}$, so the proof is complete. □

Now we are ready to prove our main theorem.

Theorem 4.14. *In the long-range correlation case 2, we have $u_\varepsilon(t,x) \Rightarrow u_0(t,x)$ in distribution as $\varepsilon \to 0$, where u_ε, u_0 solves the following PDE respectively given the same initial condition $u_\varepsilon(0,x) = u_0(0,x) = f(x)$ for $f \in \mathcal{C}_b(\mathbb{R}^d)$:*

$$\partial_t u_\varepsilon = \frac{1}{2}\Delta u_\varepsilon + i\frac{1}{\varepsilon^{\alpha/2}}V(\frac{x}{\varepsilon})u_\varepsilon, \tag{4.45}$$

$$\partial_t u_0 = \frac{1}{2}\Delta u_0 + iV_1\sqrt{c_d}\dot{W}u_0. \tag{4.46}$$

Proof. For fixed (t,x), we write

$$Z_\varepsilon := u_\varepsilon(t,x) = \mathbb{E}_B\{f(x+B_t)\exp(i\frac{1}{\varepsilon^{\alpha/2}}\int_0^t V(\frac{x+B_s}{\varepsilon})ds)\}, \tag{4.47}$$

$$Z_0 := u_0(t,x) = \mathbb{E}_B\{f(x+B_t)\exp(iV_1\sqrt{c_d}\int_0^t\int_{\mathbb{R}^d}\delta(y-x-B_s)W(dy)ds)\}, \tag{4.48}$$

and claim that $\forall m, n \in \mathbb{N}$, $\mathbb{E}\{Z_\varepsilon^m\overline{Z_\varepsilon^n}\} \to \mathbb{E}\{Z_0^m\overline{Z_0^n}\}$. Actually, we have

$$\mathbb{E}\{Z_\varepsilon^m\overline{Z_\varepsilon^n}\} = \mathbb{E}\{\prod_{j=1}^m f(x+B_t^j)\prod_{j=m+1}^{m+n}\overline{f(x+B_t^j)}\exp(\frac{i}{\varepsilon^{\alpha/2}}\int_0^t(\sum_{j=1}^m V(\frac{x+B_t^j}{\varepsilon}) - \sum_{j=m+1}^{m+n}V(\frac{x+B_t^j}{\varepsilon}))ds)\}, \tag{4.49}$$

where $B_t^j, j = 1, \ldots, N = m+n$ are independent Brownian motions. Since $V(x)$ is stationary and all relevant functions are bounded and continuous, to prove the convergence of $\mathbb{E}\{Z_\varepsilon^m\overline{Z_\varepsilon^n}\}$, we only need to prove the convergence in distribution of

$$W_\varepsilon := \sum_{j=1}^N \alpha_j B_t^j + \sum_{j=1}^N \beta_j\frac{1}{\varepsilon^{\alpha/2}}\int_0^t V(\frac{B_t^j}{\varepsilon})ds \tag{4.50}$$

for $\alpha_j, \beta_j \in \mathbb{R}$. We write $W_\varepsilon = (i) + (ii) + (iii)$ with

$$(i) = \sum_{j=1}^{N} \alpha_j B_t^j, \tag{4.51}$$

$$(ii) = \sum_{j=1}^{N} \beta_j \frac{1}{\varepsilon^{\alpha/2}} \sum_k V_1 \eta_k p_t^j(k), \tag{4.52}$$

$$(iii) = \sum_{j=1}^{N} \beta_j \frac{1}{\varepsilon^{\alpha/2}} \sum_k (\phi(\eta_k) - V_1 \eta_k) p_t^j(k), \tag{4.53}$$

where we have defined $p_t^j(k) = \int_0^t 1_{[\frac{B_s^j}{\varepsilon}+U]=k} ds$. $(iii) \to 0$ in probability by Lemma 4.11, and for $(i) + (ii)$, we calculate

$$\mathbb{E}\{\exp(i\theta_1(i) + i\theta_2(ii))\} = \mathbb{E}\{\exp(i\theta_1 \sum_{j=1}^{N} \alpha_j B_t^j) \exp(-\frac{1}{2} V_1^2 \theta_2^2 \sum_{i,j=1}^{N} \beta_i \beta_j \frac{1}{\varepsilon^\alpha} \sum_{m,n} \rho(m-n) p_t^i(m) p_t^j(n))\}, \tag{4.54}$$

and by the same proof as in Proposition 4.12, we have

$$\frac{1}{\varepsilon^\alpha} \sum_{m,n} \rho(m-n) p_t^i(m) p_t^j(n) \to \int_0^t \int_0^t \frac{c_d}{\prod_{k=1}^d |B_k^i(s) - B_k^j(u)|^{\alpha_k}} ds du \tag{4.55}$$

almost surely. Therefore, we see that

$$(i) + (ii) \Rightarrow \sum_{j=1}^{N} \alpha_j B_t^j + V_1 \sqrt{c_d} \sum_{j=1}^{N} \beta_j \int_0^t \int_{\mathbb{R}^d} \delta(y - x - B_s^j) W(dy) ds \tag{4.56}$$

in distribution given Remark 4.7, so $W_\varepsilon \Rightarrow \sum_{j=1}^N \alpha_j B_t^j + V_1 \sqrt{c_d} \sum_{j=1}^N \beta_j \int_0^t \int_{\mathbb{R}^d} \delta(y - x - B_s^j) W(dy) ds$ in distribution. Thus the claim is proved.

Note that $|Z_\varepsilon|, |Z_0|$ are uniformly bounded, if we let $Z_\varepsilon = Z_{\varepsilon,1} + iZ_{\varepsilon,2}, Z_0 = Z_{0,1} + iZ_{0,2}$, the corresponding real and imaginary parts are uniformly bounded as well. From the fact that $\mathbb{E}\{Z_\varepsilon^m \overline{Z_\varepsilon^n}\} \to \mathbb{E}\{Z_0^m \overline{Z_0^n}\}$, we know $\forall m, n \in \mathbb{N}$, $\mathbb{E}\{Z_{\varepsilon,1}^m Z_{\varepsilon,2}^n\} \to \mathbb{E}\{Z_{0,1}^m Z_{0,2}^n\}$. So

$$\begin{aligned} \mathbb{E}\{\exp(i\theta_1 Z_{\varepsilon,1} + i\theta_2 Z_{\varepsilon,2})\} &= \sum_{k=0}^{\infty} \frac{1}{k!} \mathbb{E}\{(i\theta_1 Z_{\varepsilon,1} + i\theta_2 Z_{\varepsilon,2})^k\} \\ &\to \sum_{k=0}^{\infty} \frac{1}{k!} \mathbb{E}\{(i\theta_1 Z_{0,1} + i\theta_2 Z_{0,2})^k\} = \mathbb{E}\{\exp(i\theta_1 Z_{0,1} + i\theta_2 Z_{0,2})\}, \end{aligned} \tag{4.57}$$

which completes the proof.

□

5 Conclusions and discussions

In this paper, we have studied a type of partial differential equation with a large, highly oscillatory, random coefficient with either short- or long-range correlation in dimensions $d \geq 3$. Using a Feynman-Kac formula and weak convergence approach, we have shown that the solution to the PDE converged either to a homogenized or to a stochastic equation. The case $d = 2$ could be handled in the same way with a logarithmic correction showing up in the scaling coefficient (see, e.g., [7]). The reason for the dependence of the limiting equation on the correlation property of random potential could be explained heuristically as follows. The variance of the solution is affected by the interaction of two independent Brownian motions in the random scenery, i.e., $\int_0^t V(B_s^1)ds$ and $\int_0^t V(B_u^2)du$, and we need to estimate the term $\int_0^t \int_0^t R(B_s^1 - B_u^2)dsdu$. If the random potential is short-range correlated, $R(x)$ decays sufficiently fast, then the interaction is negligible in the asymptotic limit and the variance goes to zero. On the other hand, for the long-range correlated potential, this interaction remains non-trivial as $\varepsilon \to 0$ and the limiting equation remains stochastic.

There are some possible directions to apply and extend our results. We mention three of them here. Firstly, we could consider the parabolic Anderson model $u_t = \frac{1}{2}\Delta u + V_\varepsilon u$, in which case even weak solutions may no longer be well-defined. In the long-range correlation case, it could presumably be analyzed by the estimates in [10]. In the short-range correlation case, [2] proved a homogenization results for Gaussian potential when $t \in (0, T)$ for sufficiently small T. It is not clear that for our potentials, the solution is well-defined for all $t > 0$. If we assume $V(x)$ to be Gaussian as well, to recover part of the results in [2], together with the weak convergence results given by [7] we still have to show the uniform integrability of $\exp(\int_0^t V_\varepsilon(B_s)ds)$ to pass to the limit. This could be done by writing the exponential function as a Talor series and estimate each term. We point out that in this case, diagrammatic expansions as done in [2] could be avoided. Secondly, in the long-range case, the condition $V_1 \neq 0$ ensures that the non-Gaussian case reduces to the Gaussian case so we have Gaussian noise in the limiting SPDE. If we still consider ξ_k of the form $\Phi(\eta_k)$ for some function Φ, or another type of random field $V(x) = \Phi(v(x))$ for some function Φ and Gaussian field $v(x)$, the Hermite rank of Φ determines the weak convergence limit of $\frac{1}{\varepsilon^{\alpha/2}} \int_0^t V(\frac{B_s}{\varepsilon})ds$, and the case $V_0 = V_1 = 0, V_2 \neq 0$ would presumably produce some Rosenblatt-type noise in the limit, at least in the one-dimensional setting (see, e.g., [8]). Finally, in the intermediate settings, i.e., the long-range correlation case 1, we were only able to prove a homogenization result when ξ_k are Gaussian. At present, we cannot generalize our derivation to the case when $\xi_k = \Phi(\eta_k)$, as what has been done in long-range correlation case 2. The reason is that the scaling $1/\varepsilon$ makes all the terms in the Hermite expansion contribute in the limit and we cannot reduce it to the Gaussian case even assuming the Hermite rank of Φ to be 1. Another type of limit theorem for the sum of random variables with long-range correlation is needed.

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Appendix: technical lemmas

Lemma 5.1. $\frac{a}{Tb(b-a)}(1-e^{-bT})+\frac{b}{Ta(a-b)}(1-e^{-aT})$ *is uniformly bounded when* $a,b,T>0$.

Proof. We only have to look at $f(a,b)=\frac{a}{b(b-a)}(1-e^{-b})+\frac{b}{a(a-b)}(1-e^{-a})$, and it is straightforward to check that for fixed $a>0$, $\lim_{b\to 0}f(a,b)=-1$, $\lim_{b\to a}f(a,b)=\frac{ae^{-a}-2+2e^{-a}}{a}$ is bounded, and $\lim_{a,b\to 0}f(a,b)=-1$. Assume there exists (a_n,b_n) such that $f_n=f(a_n,b_n)\to\infty$, we know that there exists a subsequence $a_n\to\infty, b_n\to\infty$ without changing the notation. If a_n/b_n is unbounded, we get the contradiction. So we assume that $a_n/b_n\to\theta\geq 0$, if $\theta\neq 1$, we get the contradiction. So we assume that $a_n/b_n\to 1$, then we only have to write

$$f_n=(e^{-a_n}-1)(\frac{1}{a_n}+\frac{1}{b_n})+\frac{a_n}{b_n}\frac{e^{-a_n}-e^{-b_n}}{b_n-a_n}\to 0, \tag{5.1}$$

as $n\to\infty$ and get the contradiction. Thus, the proof is complete. □

Lemma 5.2. *Let* $V(x)$ *be mean zero stationary random field with* $\mathbb{E}\{V(x)^6\}<\infty$*, and satisfying strongly mixing property with mixing coefficient* ϕ*, then we have*

$$|\mathbb{E}\{V(x_1)V(x_2)V(x_3)V(x_4)\}|\leq C\sum_{\{y_k\}=\{x_k\}}\phi^{\frac{1}{2}}(|y_1-y_2|)\phi^{\frac{1}{2}}(|y_3-y_4|)\mathbb{E}\{V(x)^6\}^{\frac{2}{3}}. \tag{5.2}$$

Proof. Let y_1 and y_2 be two points in $\{x_k\}_{1\leq k\leq 4}$ such that $d(y_1,y_2)\geq d(x_i,x_j)$ for all $1\leq i,j\leq 4$ and such that $d(y_1,\{y_3,y_4\})\leq d(y_2,\{y_3,y_4\})$, where $\{y_k\}_{1\leq k\leq 4}=\{x_k\}_{1\leq k\leq 4}$. We assume $d(y_3,y_1)\leq d(y_4,y_1)$. Therefore by (2.9), we have

$$\mathcal{E}:=|\mathbb{E}\{V(x_1)V(x_2)V(x_3)V(x_4)\}|\lesssim\phi(2|y_1-y_3|)(\mathbb{E}\{V(y_1)^2\})^{\frac{1}{2}}(\mathbb{E}\{(V(y_2)V(y_3)V(y_4))^2\})^{\frac{1}{2}}. \tag{5.3}$$

The last two terms are bounded by $\mathbb{E}\{V(x)^6\}^{\frac{1}{6}}$ and $\mathbb{E}\{V(x)^6\}^{\frac{1}{2}}$ respectively. Because $\phi(r)$ is decaying in $(0,\infty)$, we have $\mathcal{E}\lesssim\phi(|y_1-y_3|)\mathbb{E}\{V(x)^6\}^{\frac{2}{3}}$. On the other hand, if y_4 is (one of) the closest point(s) to y_2, the same argument shows that $\mathcal{E}\lesssim\phi(|y_2-y_4|)\mathbb{E}\{V(x)^6\}^{\frac{2}{3}}$. Otherwise, y_3 is the closest point to y_2, and we find $\mathcal{E}\lesssim\phi(2|y_2-y_3|)\mathbb{E}\{V(x)^6\}^{\frac{2}{3}}$. However, by construction, we have

$$|y_2-y_4|\leq|y_1-y_2|\leq|y_1-y_3|+|y_2-y_3|\leq 2|y_2-y_3|, \tag{5.4}$$

so we still have $\mathcal{E}\lesssim\phi(|y_2-y_4|)\mathbb{E}\{V(x)^6\}^{\frac{2}{3}}$. To summarize, we have

$$\mathcal{E}\lesssim\phi^{\frac{1}{2}}(|y_1-y_3|)\phi^{\frac{1}{2}}(|y_2-y_4|)\mathbb{E}\{V(x)^6\}^{\frac{2}{3}}, \tag{5.5}$$

and this completes the proof. □

Lemma 5.3. *Assume* $\alpha,\beta\in(0,d)$ *and* $\alpha+\beta>d$*, then*

$$\int_{\mathbb{R}^d}\frac{1}{|y|^\alpha}\frac{1}{|x-y|^\beta}dy\leq C\frac{1}{|x|^{\alpha+\beta-d}} \tag{5.6}$$

for constant C *independent of* x*. Moreover, we have*

$$\int_{\mathbb{R}^d}\frac{1}{|y|^\alpha}(1\wedge\frac{1}{|x-y|^\beta})dy\leq C(1\wedge\frac{1}{|x|^{\alpha+\beta-d}}). \tag{5.7}$$

Proof. Let $\rho = |x|$, and define $B(z,r) = \{x : |x-z| \le r\}$, $(i) = \{z : |z| < |z-x|\}, (ii) = \{z : |z| \ge |z-x|\}$, we divide $\mathbb{R}^d$ into three disjoint parts, $A_1 = B(0,\rho)\bigcap(i)$, $A_2 = B(x,\rho)\bigcap(ii)$, and $A_3 = \mathbb{R}^d\backslash(A_1\bigcup A_2)$. It is straightforward to check that when $y \in A_1$, $|y-x| \ge \rho/2$, so $\int_{A_1} \frac{1}{|y|^\alpha}\frac{1}{|x-y|^\beta}dy \lesssim \int_0^\rho r^{d-1-\alpha}dr\frac{1}{\rho^\beta} \lesssim \frac{1}{\rho^{\alpha+\beta-d}}$. Similarly, we have $\int_{A_2} \frac{1}{|y|^\alpha}\frac{1}{|x-y|^\beta}dy \lesssim \frac{1}{\rho^{\alpha+\beta-d}}$. For $y \in A_3$, we check that $|y-x| \ge |y|/2$, so $\int_{A_3} \frac{1}{|y|^\alpha}\frac{1}{|x-y|^\beta}dy \lesssim \int_\rho^\infty \frac{1}{r^{\alpha+\beta-d+1}}dr \lesssim \frac{1}{\rho^{\alpha+\beta-d}}$.

For the second inequality, we only have to note that

$$\int_{\mathbb{R}^d} \frac{1}{|y|^\alpha}(1 \wedge \frac{1}{|x-y|^\beta})dy \le \int_{|y|<2} \frac{1}{|y|^\alpha}dy + \int_{|y|\ge 2} \frac{1}{|y|^\alpha}\frac{1}{|x-y|^\beta}dy \le C \tag{5.8}$$

for $|x| < 1$. the proof is complete. □

Lemma 5.4. *For $\beta > 0$,*

$$\int_{\mathbb{R}^d} (1 \wedge \frac{1}{|y|^\beta})\frac{1}{(2\pi t)^{\frac{d}{2}}}\exp(-\frac{|y|^2}{2t})dy \to 0 \tag{5.9}$$

as $t \to \infty$.

Proof. We write $\int_{\mathbb{R}^d}(1 \wedge \frac{1}{|y|^\beta})\frac{1}{(2\pi t)^{\frac{d}{2}}}\exp(-\frac{|y|^2}{2t})dy = (i) + (ii)$, where

$$(i) = \int_{|y|<1} \frac{1}{(2\pi t)^{\frac{d}{2}}}\exp(-\frac{|y|^2}{2t})dy \to 0 \tag{5.10}$$

as $t \to \infty$, and

$$\begin{aligned}(ii) =& \int_{|y|>1} \frac{1}{|y|^\beta}\frac{1}{(2\pi t)^{\frac{d}{2}}}\exp(-\frac{|y|^2}{2t})dy = \int_1^\infty r^{d-1-\beta}\frac{1}{(2\pi t)^{\frac{d}{2}}}\exp(-\frac{r^2}{2t})dr \\ =& \frac{1}{(2\pi)^{\frac{d}{2}}}2^{\frac{d-2-\beta}{2}}\frac{1}{t^{\frac{\beta}{2}}}\int_{\frac{1}{2t}}^\infty \lambda^{\frac{d-2-\beta}{2}}\exp(-\lambda)d\lambda.\end{aligned} \tag{5.11}$$

So if $\frac{d-2-\beta}{2} > -1$, i.e., $\beta < d$, $(ii) \lesssim t^{-\frac{\beta}{2}}$. If $\beta = d$, $(ii) \lesssim t^{-\frac{\beta}{2}}\ln t$. If $\beta > d$, $(ii) \lesssim t^{-\frac{d}{2}}$ from integration by parts. The proof is complete. □

Lemma 5.5. *When $\alpha \in (0,1)$, $\int_{\mathbb{R}^2} q_\varepsilon(x)q_\varepsilon(y)\frac{1}{|z+x-y|^\alpha}dxdy \to \frac{1}{|z|^\alpha}$ as $\varepsilon \to 0$ for $z \neq 0$.*

Proof. By change of variables, we write

$$\begin{aligned}\int_{\mathbb{R}^2} q_\varepsilon(x)q_\varepsilon(y)\frac{1}{|z+x-y|^\alpha}dxdy =& \int_{\mathbb{R}^2} q_\varepsilon(w+y-z)q_\varepsilon(y)\frac{1}{|w|^\alpha}dydw \\ =& \left(\int_{|w|<\frac{|z|}{2}} + \int_{|w|>\frac{|z|}{2}}\right) q_\varepsilon(w+y-z)q_\varepsilon(y)\frac{1}{|w|^\alpha}dydw \\ =& (i) + (ii),\end{aligned} \tag{5.12}$$

and since

$$(ii) = \int_{|\sqrt{\varepsilon}w+z|>\frac{|z|}{2}} q(w+y)q(y)\frac{1}{|\sqrt{\varepsilon}w+z|^\alpha}dydw, \tag{5.13}$$

by the Lebesgue dominated convergence theorem, we have $(ii) \to \frac{1}{|z|^\alpha}$ as $\varepsilon \to 0$. For (i), we write

$$(i) = \left(\int_{|w|<\frac{|z|}{2}, |y|>\frac{|z|}{4}} + \int_{|w|<\frac{|z|}{2}, |y|<\frac{|z|}{4}} \right) q_\varepsilon(w+y-z) q_\varepsilon(y) \frac{1}{|w|^\alpha} dy dw. \tag{5.14}$$

For the first term, use $q_\varepsilon(|z|/4)$ to bound $q_\varepsilon(y)$, then integrate in y, w; for the second term, use $q_\varepsilon(|z|/4)$ to bound $q_\varepsilon(w+y-z)$, then integrate in y, w. Since $q_\varepsilon(|z|/4) \to 0$ as $\varepsilon \to 0$, we have $(i) \to 0$. The proof is complete. □

Lemma 5.6. *Assume $\alpha \in (0,1)$, then $\int_{\mathbb{R}^2} q_{\varepsilon_1}(x_1+y_1) q_{\varepsilon_2}(x_2+y_2) |y_1-y_2|^{-\alpha} dy_1 dy_2 \le C|x_1-x_2|^{-\alpha}$ for some uniform constant C.*

Proof. See Lemma A.2. in [10]. □

Lemma 5.7. *When $d \ge 3$ and $\alpha \in (0,2)$,*

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^\alpha} \int_0^t \int_0^t \mathbb{P}(|B_s - B_u| \le \varepsilon) ds du = 0. \tag{5.15}$$

Proof. By explicit calculation, we have

$$\begin{aligned}
&\frac{1}{\varepsilon^\alpha} \int_0^t \int_0^t \mathbb{P}(|B_s - B_u| < \varepsilon) ds du \\
=&\frac{1}{(\pi)^{\frac{d}{2}} \varepsilon^\alpha} \int_0^t \int_{|x|<\varepsilon} \int_{\frac{|x|^2}{2s}}^\infty \lambda^{\frac{d}{2}-2} e^{-\lambda} \frac{1}{|x|^{d-2}} d\lambda dx ds \\
=&\frac{1}{(\pi)^{\frac{d}{2}} \varepsilon^\alpha} \int_0^\infty \int_{\mathbb{R}^d} \int_0^\infty 1_{|x|<\varepsilon} 1_{|x|^2<2\lambda s} 1_{s<t} \lambda^{\frac{d}{2}-2} e^{-\lambda} \frac{1}{|x|^{d-2}} d\lambda dx ds \\
=&\frac{1}{(\pi)^{\frac{d}{2}} \varepsilon^\alpha} \int_0^\infty d\lambda \int \lambda^{\frac{d}{2}-2} e^{-\lambda} \left(\lambda s 1_{\lambda < \frac{\varepsilon^2}{2s}} + \frac{1}{2} \varepsilon^2 1_{\lambda > \frac{\varepsilon^2}{2s}} \right) 1_{s<t} ds \\
=&\frac{1}{(\pi)^{\frac{d}{2}} \varepsilon^\alpha} \int_0^\infty d\lambda \lambda^{\frac{d}{2}-2} e^{-\lambda} \left(\frac{\lambda t^2}{2} 1_{\frac{\varepsilon^2}{2\lambda} > t} + \frac{\varepsilon^2 t}{2} 1_{\frac{\varepsilon^2}{2\lambda} < t} - \frac{\varepsilon^4}{8\lambda} 1_{\frac{\varepsilon^2}{2\lambda} < t} \right) = (i) + (ii) + (iii).
\end{aligned} \tag{5.16}$$

We check that $(i) \sim \varepsilon^{d-\alpha}$, and $(ii) \sim \varepsilon^{2-\alpha}$, $(iii) \sim \varepsilon^{4-\alpha} + \varepsilon^{d-\alpha}$, so the proof is complete. □

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